ELEMENTS OF EQUILIBRIUM METHODS FOR

SOCIAL ANALYSIS

M. R. Sertel

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ELEMENTS OF EQUILIBRIUM METHODS

FOR SOCIAL ANALYSIS

by

MURAT RAUF SERTEL

B.A.(Econ) Robert College

B.Phil(Econ) Oxford University

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Signature of Author..
                Alfred P. Sloan School of Management, January 22,1971

Certified by...
                                      Thesis Supervisor

Accepted by...
           Chairman, Departmental Committee on Graduate Students.

ABSTRACT

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Murat R. Sertel

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The study is introduced by a first chapter.

The second chapter, "Formal Elements of Social Systems", first gives set-theoretical definitions for abstract social systems and their elements, establishing some terminology and notation as well. It is then discussed how a social system is a generalization of an economy and how the latter is, in turn, a generalization of a game.

Chapter 3, "Topological Foundations of Social Systems," develops some continuity and convexity results for behavors in static and dynamic social systems – all of which are defined in the previous chapter – after presenting some mathematics which is of special use in social analysis. This mathematics includes some facts concerning hyperspaces, a treatment of semi-linear topological spaces and their fixed point properties as investigated by Prakash and Sertel, and some further facts relating to the continuity and convexity properties of objective functionals and their associated infimum and supremum functionals, dealing with feasible regions as points in suitable hyperspaces.

Chapter 4, "Evolution and Equilibrium in Social Systems", first discusses some notions related to social equilibria, including Nash, Pareto, and core points, and then demonstrates existence results for social equilibrium for static and dynamic social systems. The contractual set i.e., set of social equilibria, is proved to be non-vacuous for a type of static social system and four types of dynamic social system. In the static case, Fan's fixed point theorem is applied. In the four dynamic types of social system, a more powerful theorem is needed, as a fixed point is sought in a semi-linear space. A fixed point theorem of Prakash and Sertel fits the specification and is applied. In all the cases where the contractual set is shown to be non-empty, it is shown to be compact as well. For certain social systems the contractual set is shown also

to be <u>convex</u>.

The fifth and final chapter discusses "Extensions and Applications" of the framework and theory above. The first extension indicated is that of <u>probabilistic social systems</u>. For these, a notion of a <u>behavior</u> as a probability measure on a <u>sigma-field of actions</u> is offered, matters pertaining to the <u>measurable numerical representability of preferences</u> settled, and a notion of probabilistic social system formalized. Second, as an application, a framework for the analysis of <u>power</u> is suggested, after a certain <u>causal</u> relation of an event <u>inducing</u> another is formally introduced. The resulting concept and <u>measure</u> of power is presented as a corrected generalization and formalization of Dahl's concept and measure of power. The importance of equilibrium methods for power analysis based on the above is clarified. Third, it is indicated how social systems may be viewed as <u>evolutionary systems</u>, modifying the notion of <u>dynamical system</u>, so that the <u>attraction</u> and <u>stability</u> of <u>contractual sets</u> and <u>cores</u> may be investigated. Finally, the large topic of the guidance of social systems and <u>organizations</u> via <u>incentive schemes</u>, <u>information systems</u> and other means is discussed as an area of application, suggesting also a number of extensions which promise use in the area of <u>legislation</u> and the analysis of <u>multi-level social systems</u>.

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TABLE OF CONTENTS

1. <u>INTRODUCTION</u>

This study is motivated by the belief that notions and methods of equilibrium can be developed and applied in social analysis. By social analysis is meant the positive and normative study of social systems and social phenomena.

The first thing to do was to clarify and reduce to a few the notions needed, assembling them in a consistent framework containing all that is needed and nothing that is not. The key notion here is that os a social system. For clarity and purity, it is defined set-theoretically. Next it is equipped with certain topological properties.

To do anything with the notion of equilibrium for social systems, it has to be demonstrated that such a thing exists, and that it exists for a wide enough class of social systems. This is done. Extensions and applications are discussed.

Since each chapter, except the present, begins with an introduction, the reader will be spared a lengthy introduction here.

2. FORMAL ELEMENTS OF SOCIAL SYSTEMS

The present chapter first gives definitions for social
systems and their elements, also introducing some basic terminology
and notation to be used in the sequel. The formal notion of a
'social system' as defined is then compared with the more familiar
'economy' and 'game', in both of which the existence and various
optimality and stability properties of equilibrium have been
investigated extensively and equilibrium methods of analysis
have long been used fruitfully. This comparison, from which
games emerge as restricted versions of economies and the latter
emerge as restricted versions of social systems, is intended to
provide perspective and context within which to evaluate the
framework and results presented in this study.

2.1 Preliminaries

This section is designed to introduce the basic notions and
some of the notation and terminology to be used in the sequel.
The central notion is that of a social system, defined set-
theoretically in 2.1.2 after some notation is established in 2.1.1.
Some terminology and further notation is established in 2.1.3
to refer to the elements of social systems and to some important
formal objects which are derived from these elements. The discussion
in 2.1.4 turns from matters of definition and denotation to the
intended connotation of the terminology introduced, so as to
provide some intuitive grounding for the reader's formal under-
standing. This is done by examining the typical manner in which
the elements of a social system operate. In this way it is
hoped also to communicate the motivation for the way in which
those elements are named. A motivation for the next definition,
2.1.5, distinguishing between static and dynamic social systems,
is also derived from that discussion. The consistency
of the definition of social systems is checked by 2.1.7, after
some requisite topological conventions and terminology are
established in 2.1.6. Finally, 2.1.8 adds a note to clarify
the important notion of 'incentive' in social systems.

2.1.1   Standing Notation: The empty set will be denoted by ϕ.
The set of real numbers will be denoted by R.
For any set X, [X] will denote the set of non-empty subsets
of X. Whenever [X] is endowed with a topology, that topology

will at least be as fine as the upper semi-finite (usf) topology.
(For these notions, see 3.1.2 or the classic work of Michael
(1951) concerned with topologies on spaces of subsets.)

The rather usual symbols 'Π' and 'π' will be used to
denote, respectively, products and projections. As regards
products, an important word of caution is due. Whenever the
product is the Cartesian product of mere sets, the product is to
be understood merely as such. Whenever it is a product of
topologized sets, the product is to be understood as equipped
with the product topology. Whenever it is the product of
sigma-fields, it is to be understood as the product sigma-field,
and similarly for measure spaces. When the index set is not
very crowded, the symbol '$\times$' will also be used for products.
E.g., if $\{X_i | i \in I\}$ is a family of sets indexed by $i \in I$, and
if the index set $I = \{1,2\}$, then the product may be written
as $X_1 \times X_2$, rather than $\Pi_I \{X_i\}$ or $\Pi\{X_i | i \in I\}$ or $\Pi_{i \in I} X_i$, all
of which denote the same. As a special kind of product, Y^X
will denote the set of all functions $f : X \to Y$ mapping X into Y,
i.e. such that $f(x) \in Y (x \in X)$. (Of course, Y^X is the same as
$\Pi_X \{Y_x | Y_x = Y\}$.) Finally, subscripts of π will indicate the
range of the function, so that, e.g., π_X denotes projection into X.

2.1.2 <u>Definition</u> :   A <u>social system</u> is an ordered seven-tuplet

$S = <W, U, H, G, I, T, A>$, where

(2.1.2.1)

$W = \{X_\alpha \neq \phi \mid \alpha \in A\} \neq \phi$

is a non-empty family of non-empty sets X_α,

from which we define $X = \underset{A}{\Pi} X_\alpha$, $X^\alpha = \underset{A\,\{x\}}{\Pi} X_\beta$:

(2.1.2.2)

$U = \{u_\alpha : X \times R \to R \mid \alpha \in A\}$

is an associated family of real-valued functions

u_α on $X \times R$; (see also 2.1.8);

(2.1.2.3)

$H = \{h_\alpha : X^\alpha \to X^\alpha \mid \alpha \in A\}$

is an associated family of transformations h_α,

of X^α;

(2.1.2.4)

$G = \{g_\alpha : X \to R \mid \alpha \in A\}$

is an associated family of real-valued functions

g_α on X:

(2.1.2.5)

$I = \{i_\alpha : R^X \to R^X \mid \alpha \in A\}$

is an associated family of function-valued

functions i_α assigning a real-valued function on

X to each real-valued function on X:

(2.1.2.6)
$$T = \{t_\alpha : X \times \underset{A}{\Pi D}_\alpha \to D_\alpha \subset [X_\alpha] \mid D_\alpha \neq \phi, \alpha \in A\}$$

is an associated family of functions t_α assigning

a non-empty subset of X_α to each ordered pair

whose first element is a point $x \in X$ and second

element belongs to the product $\underset{A}{\Pi D}_\alpha$ of a certain

family $\{D_\alpha \mid \alpha \in A\}$ of non-empty collections

$$D_\alpha \subset [X_\alpha];$$

(2.1.2.7)
$$A = \{\alpha : X^\alpha \times D_\alpha \to \underset{d_\alpha \in D_\alpha}{\cup} [d_\alpha] \mid \alpha \in A\}.$$

is a self-indexed family of mappings

(2.1.2.7')
$$\alpha(x^\alpha, d_\alpha) = \{x_\alpha \in d_\alpha \mid \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \underset{y_\alpha \in d_\alpha}{\text{Sup}} \ddot{w}_\alpha(y_\alpha, x^\alpha)\},$$

$$(x^\alpha \in X^\alpha, d_\alpha \in D_\alpha),$$

where $\ddot{w}_\alpha$ is defined as in 2.1.3.10 below.

2.1.3 <u>Standing Terminology and Notation</u>: Let S be as in 2.1.2.

(2.1.3.1)
X_α will be called the <u>behavior space</u> of α,

and x_α will be called a <u>behavior</u> of α

iff $x_\alpha \in X_\alpha$.

(2.1.3.2) X^α will be called the α-exclusive behavior

space of S (or of A), and x_α will be called

an α-exclusive behavior of S (or of A) iff

$x^\alpha \in X^\alpha$.

(2.1.3.3)   X will be called the (collective) behavior space

of S (or of A), and x will be called a (collective)

behavior of S (or of A) iff $x \in X$.

(2.1.3.4 u_α will be called the utility function of α.

The set R^A will sometimes be called the

distribution space of S (or of A), generic

elements $\Pi_A \{\rho_\alpha\} \in R^A$ being denoted by ρ, so

that $\pi_{R_\alpha} (\rho) = \rho_\alpha$, with $R^A = \Pi_A \{R_\alpha | R_\alpha = R, \alpha \in A\}$.

In this case, a point $\rho \in R^A$ will be called a

distribution to A and ρ_α the share of α in ρ.

The function u: $X \times R^A \to R^A$, defined by u(x,ρ)

= $\Pi_A \{u_\alpha (x, \rho_\alpha)\} (x \in X, \rho \in R^A)$ will be called

the utility scheme of S (or of A).

(2.1.3.5) h_α will be called the impression function

of α. The function h: $X \to \Pi_A X^\alpha$, defined by

$h(x) = \Pi_A \{h_\alpha (\pi_{X^\alpha} (x))\} (x \in X)$ will be called

the impression scheme of S (or of A).

(2.1.3.6) g_α will be called the <u>incentive function</u>

for α. The function $g: X \to R^A$, defined by

$g(x) = \underset{A}{\Pi}\{g_\alpha(x)\}$ $(x \in X)$ will be called the

<u>incentive scheme</u> for (or of) S (or for A).

(2.1.3.7) i_α will be called the <u>interpretation function</u>

of α The function $i: (R^X)^A \to (R^X)^A$, defined

by $i(g) = \underset{A}{\Pi}\{i_\alpha(g_\alpha)\}$ $(g \in (R^X)^A)$, will be

called the <u>interpretation scheme</u> of S (or of A).

(2.1.3.8) t_α will be called the <u>feasibility transformation</u>

for α, D_α being called the <u>feasibility space of</u> α

and a subset d_α being called a <u>feasibility</u>

for α iff $d_\alpha \in D_\alpha$. The function t, defined by

$t(x,d) = \underset{A}{\Pi}\{t_\alpha(x,d)\}$ $(x \in X; d = \underset{A}{\Pi}\{d_\alpha\}, d_\alpha \in D_\alpha,$

$\alpha \in A)$, will be called the <u>collective feasibility</u>

<u>transformation</u> for (or of) S (or for A).

(2.1.3.9)   A will be called the <u>personnel</u> of S, each member

$\alpha \in A$ being called a <u>behavor</u> .

(2.1.3.10)   For various abbreviations, the following

alternate notations will be used:

$$i_\alpha(g_\alpha) = \ddot{g}_\alpha,$$

$$u_\alpha(x_\alpha, h_\alpha(y^\alpha), i_\alpha(g_\alpha)(x_\alpha, h_\alpha(x^\alpha)))$$

$$= \ddot{u}_\alpha(x_\alpha, h_\alpha(y^\alpha), \ddot{g}_\alpha(x_\alpha, h_\alpha(x^\alpha)))$$

$$= \ddot{v}_\alpha(x_\alpha, h_\alpha(x^\alpha))$$

$$= \ddot{w}_\alpha(x_\alpha, x^\alpha).$$

The derived function $\ddot{w}_\alpha$ will be called the effective utility function of α.

2.1.4 <u>Discussion</u>: Although the rigorous development of formal results may necessitate the use of uncommon terminology and symbolism, it is difficult to overemphasize the usefulness of being able to express the underlying postulates and emerging results of a theory in intuitively pleasing fashion, reasonably within common language. The difficulties in achieving this sort of a restriction to the simple and plain, of course, are the food on which technical jargon and alienated scholarism thrive.

Realizing that there is no scarcity of jargon, especially in the class of disciplines concerned with social phenomena, the aim in term-coining here cannot be to expand the present glossary, as it would be foolish to wish to irrigate the sea. What the aim is can be expressed in two components: first, to convey the meaning of the formal framework and theory in

reasonably common and unartificial terminology, so that it is easily understood, at least in its broad outline; and, second to offer in this terminology some precision and conciseness for what are the essential elements of the framework and theory, thus hoping to bring attention to what is a small class of important elements, while giving that attention a clear focus by eliminating ambiguity and vagueness via the formalism of definition.

Now that the basic elements of a social system have been defined and a long list of terminology and notation has been introduced, some elucidation might be gained by turning to the connotation of the terms above. What the formalism of a social system S in 2.1.2-3 roughly amounts to can be expressed in plain language by describing the personnel A and the typical feasibility transformation t_α.

The personnel A can be understood in terms of its typical member α. The typical behavor α has its individual "tastes", which are in the form of a (complete) preference ordering represented (in order-preserving fashion) by the utility function u_α . In general, the utility achieved by α depends on both its own behavior x_α and the (α-exclusive) behavior x^α of all others in A. Furthermore, it may depend on a real number $\rho_\alpha \ \varepsilon \ R$, where ρ_α is to be understood as either income or wealth, or status, prestige, power or any composite of things such as these which can be expressed suitably in real numbers. The

operation of α consists of choosing a behavior, i.e., behaving, so as to optimize subject to the constraint of its feasibility and subject to its "perception" of the circumstances. This "perception" is expressed summarily by the impression function h_α and the interpretation function i_α. "Observing" a collective behavior $x \in X$, it is assumed that α "sees" the component $\pi_{X_\alpha}(x) = x_\alpha$ of x pertaining to itself as is. Although the component pertaining to the rest of A is $\pi_{X_\alpha}(x) = x^\alpha$, however, α's impression $h_\alpha(x^\alpha)$ may very well be different from x^α. Just as the equality $h_\alpha(x^\alpha) = x^\alpha$ need not hold, neither need the equality $i_\alpha(g_\alpha) = g_\alpha$ be satisfied. Thus α may "interpret" the incentive function g_α to be some different incentive function $\ddot{g}_\alpha \neq g_\alpha$. Now x^α, u_α, h_α, g_α, and i_α all influence the choice x_α made by α. Given an α-exclusive behavior x^α, α forms an impression $h_\alpha(x^\alpha) = y^\alpha$. Having interpreted g_α as $i_\alpha(g_\alpha) = \ddot{g}_\alpha$, the real number $r_\alpha = \ddot{g}_\alpha(x_\alpha, y^\alpha)$ is understood to be forthcoming as a function of the choice x_α. Thus, $u_\alpha(x_\alpha, y^\alpha, r_\alpha)$ depends on this choice, both directly and through r_α. This partly defines the optimization problem for which α is to compute a solution.

The problem to be solved by α is, in general, one of <u>constrained</u> optimization. That is to say, apart from the fact that y^α is now fixed, already imposing restraints on the values that can be taken by u_α in the present computations according

to that y^α, the choice of behavior x_α is constrained to be within a certain set $d_\alpha \subset X_\alpha$. This set, called a feasibility, is determined by the feasibility transformation t_α . The operation of α is completely described by saying that it identifies the set $\alpha(x^\alpha, d_\alpha)$ of behaviors $x_\alpha \varepsilon d_\alpha$ which maximize $u_\alpha(\cdot, y^\alpha, g_\alpha(., y^\alpha))$ on d_α. (If S is well-defined, then $\alpha(x^\alpha, d_\alpha) \neq \phi$.) Exactly one of these "best" behaviors $x_\alpha \varepsilon \alpha(x^\alpha, d_\alpha)$ is chosen, it being immaterial to α — and to us — which particular one it is.

The fashion in which the feasibility transformations operate can be seen by assuming that a feasibility d_α is given for each $\alpha \varepsilon A$ and that each α chooses a behavior $x_\alpha \varepsilon d_\alpha$ in the manner already described. The collective behavior $x = \Pi\{x_\alpha\}$ A arising in this way will, in general, now alter each feasibility in the fashion described by t_α . Specifically, each d_α is now transformed into $t_\alpha(x, d)$, where $d = \Pi\{d_\beta | \beta \varepsilon A\} \varepsilon \Pi D_\beta$ represents A A the family of feasibilities, including that (d_α) of α, which , as constraints, had governed the choice of x. In general, the equality $t_\alpha(x,d) = d_\alpha$ does not hold, t_α yielding certain behaviors $x_\alpha \varepsilon d_\alpha$ no longer feasible for α, while bringing some behaviors $z_\alpha \not\varepsilon d_\alpha$ into the new feasibility as elements $z_\alpha \varepsilon t_\alpha(x,d)$.

The operation of all of the feasibility transformations is summarized in that of the collective feasibility transformation t. Given a collective feasibility $\Pi\{d_\alpha \varepsilon D_\alpha | \alpha \varepsilon A\} = d \varepsilon D = \Pi D_\alpha$ A A

which consists of collective behaviors $z \epsilon X$ such that $\pi_{X_\alpha}(z) \epsilon d_\alpha$

$(\alpha \epsilon A)$, and given a collective behavior x such that

$\pi_{X_\alpha}(x) \epsilon d_\alpha (\alpha \epsilon A)$, $t(x,d) = \prod_A \{t_\alpha(x,d) | \alpha \epsilon A\}$.

Thus, to summarize in natural language, a social system

is a collection of behavors, each seeking its self-interest

subject to an incentive function and guided by its individual

preferences, by its interpretation of its incentive function

and by its impression of the others' behavior, and subject

also to a feasibility – which feasibility, in turn is

influenced by a history of past (collective) feasibilities and

(collective) behaviors chosen within these past feasibilities.

The remarks so far in explaining the operation of

feasibility transformations should yield the motivation for the

following definition, as well as the definition itself, rather

clear.

2.1.5 Definition: Let S be a social system and t the collective

feasibility transformation of S. S will be called (a) static

(social system) iff t is a constant map, i.e., $t(x,d) = d$ for all

$x \epsilon X$ and $d \epsilon D = \prod_A D_\alpha$. S will be called (a) dynamic (social

system) iff S is not static.

It is important to know that 2.1.2 is not a self-contradiction,

so that there exists an ordered seven-tuplet S satisfying the

definition of social system. Although that may be obvious, it

is also important to know a reasonably unrestrictive sufficient

condition for S to exist. Such a condition will be given immediatly

the following conventions are agreed upon.

2.1.6   Standing Topological Conventions and Terminology: Whenever

R is considered as a topological space, it will be assumed

to have the order topology of the natural order of real

numbers. (Recall that this is the same as the Euclidean

topology for R.)

Following Bourbaki [1966], a topological space will be

called quasi-compact iff the Borel-Lebesque condition is

satisfied, i.e., every open cover has a finite subcover.

A topological space will be compact if it is quasi-compact

and Hausdorff.

A real-valued function u: X → R on a topological space X

will be called upper semi-continuous (usc) iff $u^{-1}(\{r \in R | r \geq b\})$

is closed for all b ε R; it will be called lower semi-continuous

(lsc) iff $u^{-1}(\{r \in R | r \leq b\})$ is closed for all b ε R.

A point-to-set mapping F:X → Y of a topological space X

into a topological space Y will be called upper semi-continuous

(usc) iff F:X → [Y] is continuous with the upper semi-finite

topology on [Y](see 3.1.2 or [Michael, 1951]). Thus, F is

usc iff for each x ε X, and for each neighborhood (nbd) V of F(x),

there exists a nbd U of x such that $F(U) \subset V$.

2.1.7 <u>Proposition (Existence of S)</u>: With reference to 2.1.2–3,

each $\alpha \in A$ (hence, A; and hence, S itself) is well-defined (i.e.,

S exists as a social system), if for each $\alpha \in A$, $d_\alpha \in D_\alpha$ and

$y^\alpha \in h_\alpha(X^\alpha)$, d_α is quasi-compact and $\ddot{v}_\alpha$ is upper semi-continuous

on $\{y^\alpha\} \times d_\alpha$.

<u>Proof:</u> Assume that the hypothesis is satisfied. Clearly, all

that needs to be shown is that $\alpha(x^\alpha, d_\alpha) \neq \phi$ ($x^\alpha \in X^\alpha$, $d_\alpha \in D_\alpha$,

$\alpha \in A$). Denote $y^\alpha = h_\alpha(x^\alpha)$. Since d_α is quasi-compact, so is

$\{y^\alpha\} \times d_\alpha$. By upper semi-continuity of $\ddot{v}_\alpha$ on $\{y^\alpha\} \times d_\alpha$, $\ddot{v}_\alpha$ attains

a supremum on $\{y^\alpha\} \times d_\alpha$. Hence, $\ddot{w}_\alpha$ attains a supremum on $\{x^\alpha\} \times d_\alpha$.

Thus, $\alpha(x^\alpha, d_\alpha) \neq \phi$, as to be shown.

#

2.1.8 <u>Note (Preference and Incentives)</u>: As remarked in 2.1.4,

the utility functions u_α are meant to be order-preserving

representations of complete (preference) orders of the

behavors α on $X \times R$. Meanwhile, the functions g_α have been

called "incentive" functions to the effect that the real

numbers $r \in R$ serve to order the incentives, indicated as the

values taken by any g_α. To justify this usage of "incentive"

it is assumed from here on that, for any $x \in X$ and any $\alpha \in A$,

if r and s are real numbers such that $r \leq s$, then $u_\alpha(x,r) \leq u_\alpha(x,s)$.

This is to say that, <u>ceteris paribus</u>, a behavor does not prefer

less of the real-valued variable (incentive) to more, without

implying that more is actually preferred to less.

2.2 Games, Economies and Social Systems

Having introduced the formal notion of a social system, it is appropriate now to compare this with the more familiar notions of an economy and a game. This will give a perspective within which the place of the present study might better be judged.

The notion of an economy which will be used in this comparison is that of Arrow and Debreu (1954), although it will be presented within the present terminology and notation. This is proper enough procedure, for it will turn out that an economy — or an "abstract economy" as Arrow and Debreu called it — is a special case of a social system and that a game is a special case yet of an economy. All this will be very clear as soon as the defintions are given.

2.2.1 Definition: An economy is an ordered quadruplet

$$S = <W, \ddot{U}, \hat{T}, A>, \text{ where}$$

(2.2.1.1) W is as in 2.1.2.1; from which X and X^{α} are defined as there;

(2.2.1.2) $\ddot{U} = \{\ddot{u}_{\alpha} : X \to R \mid \alpha \in A\}$

is an associated family of real-valued functions $\ddot{u}_{\alpha}$ defined on X;

(2.2.1.3) $\hat{T} = \{\hat{t}_{\alpha} : X^{\alpha} \to [X_{\alpha}] \mid \alpha \in A\}$

is an associated family of mappings $\hat{t}_\alpha$ assigning

a non-empty subset $\hat{t}_\alpha(x^\alpha)$ of X_α to each $x^\alpha \in X^\alpha$:

(2.2.1.4) $\quad A = \{\alpha : X^\alpha \times \hat{t}_\alpha(X^\alpha) \to [X_\alpha] \mid \alpha \in A\}$

is a self indexed family of mappings defined by

$$\alpha(x^\alpha, d_\alpha) = \{x_\alpha \in d_\alpha \mid \ddot{u}_\alpha(x_\alpha, x^\alpha) \geq \underset{y_\alpha \in d_\alpha}{\text{Sup}} \; \ddot{u}_\alpha(y_\alpha, x^\alpha)\},$$

$$(x^\alpha \in X^\alpha, \; d_\alpha \in \hat{t}_\alpha(X^\alpha)).$$

2.2.2 $\quad$ <u>Definition</u>: A <u>game</u> is an economy in which $\hat{t}_\alpha(x^\alpha) = X_\alpha$ for

all $x^\alpha \in X^\alpha$ and $\alpha \in A$.

2.2.3 $\quad$ <u>Note</u>: The actual definitions of Arrow and Debreu (1954) from

which 2.2.1-2 is generalized actually has A as a finite set,

but given more recent developments (Aumann, 1964) in which

a continuum of traders (players) is considered, it is unreasonable

to stick to such a restriction - a restriction which is

unnecessary in the first place, except possibly to yield

economics understandable with the tools of Euclidean space.

A simple comparison of 2.2.1-2 with 2.1.2 yields that, indeed, an

economy is a special sort of social system and that a game is a

special case of an economy. This notion of a game may or may not

be appropriate. It does have the authorization of Arrow and Debreu,

however, and that should carry some weight. In any case the above

defintions 2.2.1-2 will not be used to derive any results of this

study, but are recorded merely for the comparison they allow.

Coming to that comparison, it is to be noted firstly that the

elements H, G and I of a social system have been suppressed in

2.2.1. Hence, the functions $\ddot{u}_\alpha$, as is intended to be suggested by

using the notation of umlaut ('¨'), are analogous to the effective

utility functions $\ddot{w}_\alpha$ of 2.1.3.10, but not necessarily such, since

they are not explicitly derived from functions u_α, h_α, g_α and i_α as are

the functions $\ddot{w}_\alpha$. Finally, the mappings $\hat{t}_\alpha$ of 2.2.1.3 are restricted

versions of the feasibility transformations t_α of 2.1.2.6 and 2.1.3.8.

The $\hat{t}_\alpha$'s depend only on α-exclusive behaviors x^α. The t_α 's

are allowed to depend, in addition, on the behavior x_α of α as well

as the collective feasibility d in $D = \Pi D_\alpha$.
                                              A
All this being so, the restrictions of 2.2.1 may be viewed in

different lights. Accordingly, one view might be that the

economist is not interested in the details of the full-blown social

system and it is a useful simplification to suppress the perceptive-

cognitive and information-systemic elements for which H and I stand and

that the incentive scheme - which is associated with G - does not

matter. Before turning to the simplification of the feasibility

transformations t_α to the form $\hat{t}_\alpha$, it is worth challenging the above

view. For to say that economists are not concerned with incentives

wouldbe to say that they are not concerned with prices - wages included -

or with taxes and subsidies – incentives for investment, etc., included. And to say that they are not interested in the effects of imperfections in information or in its processing by the user, e.g., in the "marketplace", would imply statements to the effects, for instance, that a devaluation can be announced a week earlier than it is consummated or that advertising has yet to be invented. It is very difficult, therefore, to defend that ignorning the elements H, G and I is a useful simplification or idealization in the genre of the "ideal gas" or the "billiard ball" model of gasses.

The simplification of the feasibility transformations t_α to the form $\hat{t}_\alpha$ is also a difficult one to defend. For one thing, what is feasible for an economic agent obviously depends on the behavior of that agent itself. It would require some rather strong metaphysics otherwise to explain why people or firms or governments choose to save and invest if tomorrow's vacations, factories and parks did not depend on whether one saved a penny or built a factory or upkept a park today. Secondly, the factories one has tomorrow depends on what factories one has today. For instance, one allows the textile industry to slowly depreciate its equipment and invests in the electronics industry. The production possibilities set of tomorrow depends on the present one and on the point now chosen in it. That is to say, the feasibility for the agent α depends on x_α and d_α.

To include certain external effects, it is here permitted to depend on the feasibilities d_β of the other agents $\beta \in A$ as well. (To give a possibly odd example for where this may become relevant, it might be considered that the precedence set by <u>allowing</u> one boy to be a conscientious objector to war, <u>whether or not</u> the boy uses this priviledge, will probably make it easier for the next boy to gain this choice).

The result of the above discussion seems to be not really that an economy <u>is</u> a special case of a social system, but that this would be so if one went by the definitions which were compared. But the result is also that this would be a very artificial exercise of classification and that a social system, as defined by 2.1.2 is really something dear to the interests of economics. In its formal specification, nevertheless, it is more general than the economy for which Arrow and Debreu proved the existence of an equilibrium. It will be found then that the equilibria proved in 4.3. to exist in the case of dynamic social systems generalize the result of Debreu (1952) obtained for a certain special class of social systems. The mentioned work of Debreu is actually the main mathematical pillar on which the outstanding Arrow and Debreu study is based, the social systems treated in it being correspondingly specialized. (Typically, both works deal with a finite personnel and with behaviors in Euclidean space, and these constitute a further restriction on their results. Although a preference to work in such spaces is often thought

to be"realistic" (perhaps because it is "less abstract"), as far as realism is concerned, any result which is true with weaker assumptions is at least as realistic as the same result with stronger assumptions. Furthermore, it is not possible to represent, for example, an infinite-horizon plan naturally as a point in finite dimensional space, its natural habitat being infinite dimensional. Hence, for this and many other reasons, neither is it the case that all economics can be reasoned in Eucliden space.)

The most summary comparison of a social system with a game, to end this section, would be that the latter is static (see 2.1.5) Thus, the existence result of 4.2 can be regarded as a generalization of Nash's [1950, 1951] result for (again a restricted variety of) finite-personnel games to the case of a certain class (type 0) of static social systems.

3. TOPOLOGICAL FOUNDATIONS OF SOCIAL SYSTEMS

This chapter first presents some mathematics, mostly topology, which is especially useful in the analysis of social systems. In one way or another, all of this material is actually used in the present study, but much more can be expected from its use than would fit within the constraints of this investigation. Furthermore, most of the material is either new to the field of social analysis or plain new. As best as an amateur historian can do, the origins and intellectual history of the material are indicated.

Next, some fundamental topological properties of behavors - and, thus, indirectly of social systems - are demonstrated as deriving from various properties, if they pertain, of elements such as utility, impression, interpretation and incentive functions and behavior spaces. These are demonstrated first for static and then for dynamic social systems. They are used in the corresponding theories of existence for social equilibrium, presented in the next chapter.

3.1 Topological Preliminaries

This section collects some topological facts crucial to the later sections and chapters, so that they may be used freely without explicit reference once they are recorded. Most of these facts relate to hyperspaces and to real semi-linear topological spaces (rst spaces).

The idea of hyperspace, i.e., a topological space whose points are subsets of a topological space, dates at least as far back as the metric defined by Hausdorff [1937] on the set of non-empty closed subsets of a bounded metric space X. (see also [Kelley, 1942] for a study of the Hausdorff metric hyperspace when X is compact.) Meanwhile, Vietoris [1923] defined the finite topology (see 3.1.2,10) for the set of non-empty closed subsets of an arbitrary topological space X. The standard reference adopted here, however, is the complete and unifying study of Michael [1951].

The importance of hyperspaces for optimization, economic theory and social analysis in general derives from at least two considerations. One of these in turn derives from the importance of point-to-set mappings in these fields. For a point-to-set mapping, such as an optimizing algorithm, a consumer choosing bundles of goods or a behavor choosing a set of behaviors, can be looked upon as a point-to-point mapping on the same domain

of definition into a suitable hyperset (set of subsets) of the range. If
the domain and range of the point-to-set mappings are topological spaces,
then matters relating to the (upper or lower semi-) continuity of this
mapping are often simplified by appropriate choice of a topology for the
hyperset serving as range for the associated point-to-point mapping. This
is a primary use made of hyperspaces in this study, as 3.1.4.4-5 and the
application of these in 3.3.1, and thus in each of the results of 4.3,
constitute such a use.

These mentioned applications in the present study also illustrate
the second general use of hyperspaces for the mentioned fields of
inquiry. The consideration here is that "feasible regions" can be
regarded as points in a hyperspace, so that changes in these can be
analyzed by use of point-to-point mappings (and even, as in 4.3.3.3-5,
by use of point-to-set mappings) into that hyperspace. The power of such
methods will probably be felt less in optimization constrained to
feasible regions in Euclidean space (especially when the constraints
are finitely parametrized, as in the case of linear constraints of budget,
etc.), but in dynamic optimization where decisions taken are allowed to alter
the very feasible regions within which they are taken - as in the case of
(dis-)investment - and especially when the feasible regions lie in some
abstract space, such as a function space, and the constraints are
not finitely parametrizable, these methods may be expected to bear fruit.
Even a restriction of the results in 4.3 to the case of a singleton
personnel might testify to the validity of such an expectation.

Semi-linear topological spaces, and, more generally, semi-linear spaces were first investigated, to the knowledge of this author, by Prakash and Sertel [1970, a,b]. As a generalization of linear (or vector) spaces, semi-linear (or semi-vector) spaces have an algebra which is satisfied, notably for present purposes, by the set of all non-empty subsets of a vector space. In the case where L is a linear topological space, the set of non-empty quasi-compact subsets of L form a semi-linear space. Among semi-linear topological spaces, those which are used in this study are the ones formed by the Hausdorff metric space of non-empty compact and convex subsets of a normed real linear topological space. Thus, the usual feasible regions in usual constrained optimization are typical points of such a (rst) space.

3.1.1 Standing Notation: For any topological space X, C(X) will denote the set of all non-empty closed subsets of X, k(X) will denote the set of all non-empty quasi-compact subsets of X, and K(X) will denote the set of all non-empty compact subsets of X. If X is a convex set, Q(X) will denote the set of all non-empty convex subsets of X. Furthermore, CQ(X) = C(X) $\cap$ Q(X), kQ(X) = k(X) $\cap$ Q(X), etc., will also be used.

If f: X → Y is a mapping, then $\Gamma(f) = \{(x,y) \mid x \in X, y \in f(x)\} \subset X \times Y$ will be used as standard notation for the graph of f.

3.1.2 <u>Hyperspaces</u>: This section follows [Michael, 1951], extracting the bare minimum of information needed for the subsequent development.

3.1.2.1 <u>Notation</u>: Let $U \subset X$. Then denote

$$<U>^+ = \{Y\epsilon[X] | Y \subset U\},$$

$$<U>^- = \{Y\epsilon[X] | Y \cap U \neq \phi\}.$$

Let $\{U_i | i \epsilon I\}$ be a collection of subsets $U_i \subset X$. Then denote

$$<U_i | i \epsilon I> = \{Y\epsilon[X] | Y \subset \underset{I}{U}U_i, Y \cap U_i \neq \phi \text{ for all } i \epsilon I\}.$$

If I above is finite, so that $\{U_i | i \epsilon I\} = \{U_1,\ldots,U_n\}$, then also denote $<U_i | i \epsilon I>$ by $<U_1, \ldots, U_n>$.

3.1.2.2 <u>Definition</u>: Let X be a topological space with topology τ. The <u>upper semi-finite (usf) topology</u> on $[X]$ is the topology generated by $\{<U^+> | U \epsilon \tau\}$ as a basis. The <u>lower semi-finite (lsf) topology</u> on $[X]$ is the topology generated by $\{<U>^- | U \epsilon\tau\}$ as a sub-basis. The <u>finite (f) topology</u> on $[X]$ is the topology generated by $\{<U_1, \ldots, U_n> | \{U_1, \ldots, U_n\} \subset \tau\}$.

3.1.2.3   Remark: Using 'c' as a superscript to denote complements, the following equations follow easily from the definitions:

$$\text{(i)} \quad <U>^{+} = <U> = [U] = (<U^{c}>^{-})^{c},$$

$$\text{(ii)} \quad <U>^{-} = <U^{c}>^{c},$$

$$\text{(iii)} \quad <U>^{c} = <X, U^{c}>$$

The following spells out, for the benefit of the reader, a proof for a proposition observed by Michael.

3.1.2.4   Proposition: Let X be a topological space. The finite topology on [X] is the coarsest topology, in the lattice of all topologies on [X], which is finer than both the usf and the lsf topology on [X]. [Michael, 1951, p. 179].

Proof: It suffices to show that the usf and the lsf topology on [X] are contained in any topology containing the finite topology on [X]. From the first equation of 3.1.2.3 it follows that the usf topology is so contained. Let $U \subset X$ be open. Then, using the last two equations of 3.1.2.3, $<U>^{-} = <U^{c}>^{c} = <X,U>$, proving that the lsf topology is also so contained.

The purpose of the following is, again, elucidation.

3.1.2.5 Proposition: Let X be a topological space, and denote the usf, lsf and f topologies, on [X], by τ^+, τ^-, τ^{+-}, respectively. Then

(i) τ^+ is the coarsest topology on [X] for which (a) $\langle U \rangle^+$ is open if U is open in X and (b) $\langle M \rangle^-$ is closed if M is closed in X:

(ii) τ^- is the coarsest topology on [X] for which (a) $\langle U \rangle^-$ is open if U is open in X and (b) $\langle M \rangle^+$ is closed if M is closed in X:

(iii) τ^{+-} is the coarsest topology on [X] for which (a) $\langle U \rangle$ is open if U is open in X and (b) $\langle M \rangle$ is closed if M is closed in X.

Proof: It follows directly from the definitions that τ^+ and τ^- are the coarsest topologies in [X] satisfying parts (a) of (i) and (ii), respectively. In the following let $U \subset X$ be open, and w.l.g., let $M = U^c$. It suffices to show that parts (b) of (i) and (ii) hold for τ^+ and τ^-, respectively, for then (iii) will follow by 3.1.2.4. To see that (i) (b) holds for τ^+, just note that

$$\langle M \rangle^- = (\langle U \rangle^+)^c,$$

which is closed since $<U>^+$ is open. To see that (ii) (b)

holds for τ^-, note, similarly, that $<M>^+ = (<U>^-)^c$ is closed.

#

The rest of this section following the present paragraph, with the possible exception of 3.1.2.9 is merely paraphrased from [Michael, 1951]. The first definition gives a useful equivalent rewording of the usual definition of uppar and lower semi-continuity for multi-valued binary relations (point-to-set mappings). The remainder will be useful after the next section introduces a special rst space which is a hyperspace.

3.1.2.6   Definition: If X and Y are topological spaces, a mapping F: X →[Y] is called <u>upper (lower) semi-continuous (u(1)sc)</u> iff F is continuous with the u(1)sf topology on [Y]. [Michael, 1951; p. 179].

The last definition can be reworded also as follows.

3.1.2.7   Proposition: If X and Y are topological spaces, then a function F: X → [Y] is u(1) sc iff $\{x \ \varepsilon \ X | F(x) \cap A \neq \phi\}$ is closed (open) whenever A is closed (open) in Y. [Michael, 1351; Thm. 9.1]

3.1.2.8 <u>Definition</u>: Define the ("<u>layout...) map (...of X"</u>)

L:[[X]] → [X] by

$$L(\{Z_i \epsilon [X] \mid i \; \epsilon \; I \;) = \bigcup_I Z_i.$$

[Michael, 1951; Def. 5.5.1]

3.1.2.9 <u>Proposition</u>: If X is a topological space, then the layout

map of X is (i) usc, (ii) lsc and (iii) continuous accordingly

as [X] and [[X]] carry the (i) usf, (ii) lsf, and (iii) finite

topology (Cf. [Michael 1951 Thm. 5.7.2]).

<u>Proof</u>: Denote generic elements of [[X]] by E, and defin-

j ε J_E iff $Y_j \epsilon$ E, so as to be able to write E = $\{Y_j \mid j \; \epsilon \; J_E\}$.
The proofs of (i) (ii), (iii) are entirely set-theoretic.

<u>ad (i)</u>: Let V = $\langle U \rangle^+$ be a basic open nbd of Y = L(E) ε [X].
It suffices to show that $L^{-1}(V) = \langle V \rangle^+$:

$$E \; \epsilon \; L^{-1} \; (V) \; \text{iff} \; \bigcup_{J_E} Y_j = Y \; \epsilon \; V$$

$$\text{iff } Y \subset U$$

iff   ($j \in J_E$ only if $Y_j \subset U$)

iff   ($j \in J_E$ only if $Y_j \in V$)

iff $E \subset V$

iff $E' \in \langle V \rangle^+$.

ad (ii): Let $V = \langle U \rangle^-$ be a basic open nbd of $Y = L(E) \in [X]$. It suffices to claim that $L^{-1}(V) = \langle V \rangle^-$, and to show it as follows:

$E \in L^{-1}(V)$ iff $Y \in V$

iff $Y \cap U \neq \phi$

iff   ($\exists_{j_*} \in J_E$ such that $Y_{j_*} \cap U \neq \phi$)

iff   ($\exists_{j_*} \in J_E$ such that $Y_{j_*} \in V$)

iff $E \cap V \neq \phi$

iff $E \in \langle V \rangle^-$.

ad (iii): Let $V = \langle U_1, \ldots, U_n \rangle$ be a basic open nbd of $Y = L(E) \in [X]$ and denote $N = \{1, \ldots, n\}$, $U = \underset{N}{U} U_i$ and $W = \langle\langle U \rangle^+\rangle^+ \cap (\underset{N}{\cap} \langle\langle U_i \rangle^-\rangle^-)$. It suffices to show that $L^{-1}(V) = W$:

$E \in L^{-1}(V)$ iff $Y \in V$

iff $Y \subset U$ and $Y \cap U_i \neq \phi$ for all $i \varepsilon N$

iff $(j \varepsilon J_E$ and $i \varepsilon N$ only if $Y_j \subset U$ and $Y \cap U_i \neq \phi)$

iff $(j \varepsilon J_E$ and $i \varepsilon N$ only if $Y_j \varepsilon <U>^+$ and $Y\varepsilon<U_i>^-)$

iff $E \varepsilon <<U>^+>^+$ and $E \varepsilon \cap_N <U_i>^->^-$

iff $E \varepsilon W.$

3.1.2.10   <u>Notation-Definition-Remark-Proposition</u>: Replace [X] in

3.1.2.1-9 by C(X) and modify 3.1.2.3 to state that $<U> = C(U)$

if U is closed in X.

3.1.2.11   <u>Proposition</u>:

1. If X is a regular space and $E \varepsilon k(C(X))$ with the usf

topology on C(X), then $L(E) \varepsilon C(X)$;

2. If X is a topological space and $E \varepsilon k(k(X))$ with the

usf topology on k(X), then $L(E) \varepsilon k(X)$.

[Michael, 1951; Thm. 2.5.1-2, Thm. 9.5].

3.1.2.12 <u>Proposition</u>: Let X be a topological space, and let C(X)

be equipped with the finite topology. Then X is quasi-compact,

locally quasi-compact, separable, compact iff C(X) has the

same property [Michael, 1951; Thm. 4.2., Thm. 4.4.1, Thm. 4.5.1,

Thm. 4.9.6].

PLEASE INSERT

P. PRAKASH and M. R. SERTEL

"SEMILINEAR (TOPOLOGICAL) SPACES AND APPLICATIONS"

rather than the older

P. PRAKASH and M R. SERTEL

"FIXED POINT AND MINMAX THEOREMS IN SEMILINEAR SPACES"

into 3.1.3 on p. 34 of

M. R. SERTEL

ELEMENTS OF EQUILIBRIUM METHODS FOR SOCIAL ANALYSIS,

Ph.D. Dissertation, M.I.T. February, 1971

3.1.2.13 <u>Proposition</u>:   Let X be a metric space. Then the finite

topology on C(X) agrees with the Hausdorff metric topology

on C(X) iff X is (quasi-)compact. [Michael, 1951, Thm. 3.3,

Prop. 3.5].

3.1.3   <u>Rst Spaces</u>: Rather than lengthen the present chapter by para-

phrasing or reproducing, the original work by Prakash and

Sertel on semi-vector spaces, semi-linear topological spaces,

rst spaces and their fixed point properties is appended to this

study.

3.1.4   <u>Special Facts Basic to Optimization</u>: This section collects some

facts relating to the continuity and convexity matters pertaining

to functionals typically playing the role of objective functional

in optimization problems. All of these facts are well-known in

certain restricted instances, as when the functional is defined on

a subspace of Euclidean space. The novelty in the more general

facts presented here derives from the treatment of the usual

feasible regions as points in suitable spaces. The various

continuity and convexity properties of the objective functional

are related through this treatment to corresponding properties of

the optimal value attained on a feasible region, depending on the

abstract feasible region as a variable . Thus, e.g., the

supremum attained by an objective functional on, say, a compact

feasible region is seen to share much of the continuity and

convexity properties of the objective functional itself, although

the supremum depends on the feasible region while the objective

functional depends on a generic element of the space in which

such feasible regions lie as sets. Such facts are essential

when the feasible regions become endogenous variables of the model,

as in dynamic optimization or dynamic social systems.

First considered are matters of continuity. The first lemma,

3.1.4.1, plays a key role here. While 3.1.4.1-3 are concerned with

the objective functional, the two simple propositions 3.1.4.4-5

are related in an obvious way to the feasibility. Then

considered are the convexity properties, relating, again, to the

objective functional.

The main results are all concerned with how continuity,

convexity - and various weaker versions of these properties -

for the objective functional relate to corresponding properties of

the associated "supremum" or "infimum functional". The results

are presented in "disaggregated" form, that is, continuity questions

are split into questions of upper and lower semi-continuity,

and convexity or concavity matters are formulated in terms of

strict and non-strict versions and in terms of (strict and non-strict)

quasi-convexity or quasi-concavity.

3.1.4.1   LEMMA: Let B be a closed set in a compact (Hausdorff)

space X × Y.   Define

$$D = \{(k,y) \; \epsilon \; C(X) \times Y \mid (k \times \{y\}) \subset B\} \; ,$$

$$E = \{(k,y) \; \epsilon \; C(X) \times Y \mid (k \times \{y\}) \cap B \neq \phi, \}, $$

and assume that $C(X)$ has the finite topology. Then

1. D is compact in $C(X) \times Y$; and

2. E is compact in $C(X) \times Y$.

Proof: Denote $Y^* = \pi_Y(B)$, $B_y = B \cap (X \times \{y\})$, $B^*_y = \pi_X(B_y)$; it is clear that all of these are compact. Furthermore, $C(X) \times Y$ is a compact (Hausdorff) space when $C(X)$ has the finite topology, as $C(X)$ is then compact (Hausdorff). (See 3.1.2.12). Hence, it suffices to show that (1) D and (2) E are closed.

ad (1): There is nothing to prove if $D = C(X) \times Y$, so let $A \; \epsilon \; (C(X) \times Y) \backslash D$. Denote $D_y = <B^*_y> \times \{y\}$ and generic elements $d_y = (k,y) \; \epsilon \; D_y (y \; \epsilon \; Y^*)$. Then A and d_y are distinct for each $d_y \; \epsilon \; D$. Fixing $y \; \epsilon \; Y^*$, there thus exists an open cover $\{W(d_y) \mid d_y \; \epsilon \; D_y \}$ of D_y with open boxes

$$W_y(d_y) = U_y(k) \times V_y(y), \qquad (d_y \; \epsilon \; D_y),$$

and a family $\{N_y(d_y) \mid d_y \; \epsilon \; D_y\}$ of nbds $N_y(d_y) \subset C(X) \times Y$ of A, such that for each $d_y \; \epsilon \; D_y$, $W_y(d_y) \cap N_y(d_y) = \phi$ (where $U_y(k) \subset C(X)$ is an open nbd of k and $V_y(y) \subset Y$ is an open

nbd of y). Since $<B^*_y>$ and $\{y\}$ are compact, so is D_y.

Hence, $\{W_y(d_y) \mid d_y \in D_y\}$ affords a finite subcover

$$\{W_y(d_y^i) = U_y^i \times V_y^i \mid i = 1, \ldots, m(y)\}.$$

Define $V_y = V_y^1 \cap \ldots \cap V_y^{m(y)}$,

and $N_y = N_y(d_y^1) \cap, \ldots \cap N_y(d_y^{m(y)})$ and

$$W_y^i = U_y^i \times V_y \qquad (i = 1, \ldots, m(y)).$$

Then $\{W_y^i \mid i = 1, \ldots, m(y)\}$ is an open cover of D_y such that

$$W_y^i \cap N_y = \phi \qquad (i = 1, \ldots, m(y)).$$

Now $\{V_y \mid y \in Y^*\}$ is an open cover of the compact Y^*,

affording a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Define

$N = \bigcap_{j=1}^{n} N_{y_j}$. Then N is a nbd of A which is disjoint from

$$W = \bigcup \{W_{y_j}^i \mid i = 1, \ldots, m(y_j);\ j = 1, \ldots, n\}.$$

Since $D \subset W$, $N \cap D = \phi$. Thus, D is closed, since its complement in $C(X) \times Y$ is open.

<u>ad(2)</u>: To show that E is closed, note that

$$E = \bigcup_{Y^*} (<(B_y^*)^c>^+)^c \times \{y\}$$

and that $E_y = (<(B_y^*)^c>^+)^c \times \{y\} = <B_y^*>^- \times \{y\}$ is closed, hence

compact, for each $y \in Y^*$. Then the proof of (1) applies, merely

by replacing $<B_y^*>$ by $<B_y^*>^-$, D_y by E_y, D by E and d_y by e_y.

#

3.1.4.2 <u>Lemma</u>: Let $X = X_\alpha \times X^\alpha$ be a compact (Hausdorff) space, let $u: X \to R$ be continuous, and define the real-valued functions $\bar{u}$, $\underline{u}$ on $C(X_\alpha) \times X^\alpha$ by

$$\bar{u}(k_\alpha, x^\alpha) = \operatorname*{Sup}_{k_\alpha} u(\cdot, x^\alpha)$$

$$\left. \begin{array}{l} \\ \\ \\ \underline{u}(k_\alpha, x^\alpha) = \operatorname*{Inf}_{k_\alpha} u(\cdot, x^\alpha) \end{array} \right\} k_\alpha \, \varepsilon \, C(X_\alpha), \, x^\alpha \, \varepsilon \, X^\alpha \, .$$

Assume $C(X_\alpha)$ has the finite topology. Then

1.1. u is usc iff $\underline{u}$ is usc,

  2 u is lsc iff $\bar{u}$ is lsc;

2.1. u is usc iff $\bar{u}$ usc,

  2. u is lsc iff $\underline{u}$ is lsc.

<u>Proof</u>: Upon noting that X_α is T_1, it is obvious that continuity properties of $\bar{u}$ or $\underline{u}$ hold also for u, since a continuity property holding for $\bar{u}$ or $\underline{u}$ on the whole of $C(X_\alpha) \times X^\alpha$ also holds on the subspace $\{(\{x_\alpha\}, x^\alpha)\}$, while this subspace is homeomorphic to X and $u = \bar{u} = \underline{u}$ on this subspace. All the implications 'if' are thus proved.

To prove the rest, let $p \in R$ be arbitrary, and

denote $P^+ = \{r \in R | r \geq p\}$, $P^- = \{r \in R | r \leq p\}$, $u^{-1}(P^+)$

$= B^+$, $u^{-1}(P^-) = B^-$, $\bar{u}^{-1}(P^+) = E^+$, $\bar{u}^{-1}(P^-) = D^-$, $\underline{u}^{-1}(P^+) = D^+$,

$\underline{u}^{-1}(P^-) = E^-$. From this notation, it is clear how to use the

last lemma, observing that B^+ is closed (compact) if u is usc

and that B^- is closed (compact) if u is lsc. Thus, in case

of 1.1., D^+ is closed, so that u is usc; in case of 1.2, D^-

is closed, so that $\bar{u}$ is lsc; in case of 2.1, E^+ is closed, so

that $\bar{u}$ is usc; in case of 2.12, E^- is closed, so that $\underline{u}$ is lsc.

This completes the proof.

#

3.1.4.3　Corollary: Using the definanda and notation of the last

lemma, among the following statements i.a, i.b and i.c

are equivalent (i = 1 , 2, 3):

1. a. u is usc.

   b. $\underline{u}$ is usc with the finite topology on $C(X_\alpha)$.

   c $\bar{u}$ is usc with the finite topology on $C(X_\alpha)$.

2. a. u is lsc.

   b. $\underline{u}$ is lsc with the finite topology on $C(X_\alpha)$.

   c. $\bar{u}$ us lsc with the finite topology on $C(X_\alpha)$.

3. a. u is continuous.

   b. $\underline{u}$ is continuous with the finite topology on $C(X_\alpha)$.

c. $\bar{u}$ is continuous with the finite topology on $C(X_\alpha)$.

Proof: All is plain as a rearrangement of the last lemma.

#

The form in which 3.1.2.9 will actually be used (together with 3.1.4.5 in proving 3.3.1) is actually the following simple proposition, needing no proof.

3.1.4.4 Proposition: Let X be a topological space, and define the map $\ell : [X] \to X$ by

$$\ell(E) = \{x \mid x \in E\} \qquad (E \in [X]).$$

Then ℓ is usc (lsc) with the usf(lsf) topology on [X].

3.1.4.5 Proposition: The graph $\Gamma(\ell) = \{(E,x) \mid E \in C(X), x \in E\} \subset C(X) \times X$ is closed if X is regular and C(X) has the usf topology.

Proof: To see that the complement of $\Gamma(\ell)$ is open, let $F \in C(X)$ and $y \in X \setminus F$. Since F is closed and X is regular, there exist disjoint open sets U, $V \subset X$ such that $F \subset U$ and $y \in V$. Then <U> is open with $F \in <U> \subset C(X)$, and $(<U> \times V) \cap \Gamma(\ell) = \phi$, showing that $\Gamma(\ell)$ has open complement.

#

In the rest of the section attention is directed to matters of convexity of u, $\bar{u}$, and $\underline{u}$.

3.1.4.6   LEMMA: Let L_α and L^α be real linear topological spaces and, with reference to 3.1.4.2-3, let $X^\alpha \in KQ(L^\alpha)$, $X_\alpha \in KQ(L_\alpha)$, and assume u to be continuous. Then u has any of the properties under (1) below iff $\bar{u}$ has, and u has any of the properties under (2) below iff $\underline{u}$ has:

1. a (strict) quasi-concavity,

    b. (strict) concavity,

    c. linearity, i.e., concavity and convexity;

2. a. (strict) quasi-convexity,

    b. (strict) convexity,

    c. linearity.

Proof: The 'if' parts of the proposition are all obvious upon noting that singleton subsets of X_α are closed, and that the collection of these is convex. The 'only if' parts are all straightforward, so only (1.a) will be treated; imitation will yield the remaining proofs.

Let $'\lambda = 1 - \lambda \in [0,1]$, let (k_α, x^α), $('k_\alpha, 'x^\alpha) \in C(X_\alpha) \times X^\alpha$, and let $x_\alpha \in k_\alpha$ and $'x_\alpha \in 'k_\alpha$ with $u(x_\alpha, x^\alpha) = \bar{u}(k_\alpha, x^\alpha)$ and $u('x_\alpha, 'x^\alpha) = \bar{u}('k_\alpha, 'x^\alpha)$. Finally, denote $\bar{k}_\alpha = \lambda k_\alpha + '\lambda 'k_\alpha$,

$$\bar{x}^\alpha = \lambda x^\alpha + {'\lambda}{'x}^\alpha \text{ and } \bar{x}_\alpha = \lambda x_\alpha + {'\lambda}{'x}_\alpha. \text{ Then } \bar{u}\ (\bar{k}_\alpha, \bar{x}^\alpha) \geq$$

$u(\bar{x}_\alpha, \bar{x}^\alpha)$. If u is quasi-concave, then

$$u(\bar{x}_\alpha, \bar{x}^\alpha) \geq \text{Min } [u(x_\alpha, x^\alpha), u({'x}_\alpha, {'x}^\alpha)],$$

whereby $\bar{u}$ is quasi-concave also; similarly, if u is strictly

quasi-concave (i.e., the last inequality is strict), then

$\bar{u}$, too, is so.

#

3.1.4.7 <u>Remark:</u> From 3.1.4.2.-3, it is clear that 3.1.4.6 can be

-strengthened by assuming only that u is usc for part 1 and

lsc for part 2.

3.2 <u>Topological Properties of Behavors in Static Social Systems</u>

From a narrow viewpoint, the prime motivation for recording the

properties collected in this section is the existence theory for

equilibrium in static social systems, presented in 4.2. The end

of the proof for an existence theorem for social equilibirum, however,

marks just a beginning for social analysis, no matter how general

and powerful that existence theorem. The properties enjoyed (or

suffered) by behavors in a social system - static or dynamic -

deserve consideration, therefore, as main building blocks of social

analysis, rather than merely as stepping stones useful only

for proving an equilibrium existence theorem. For this reason,

the present section treats selected properties of behavors in

some detail, restricting attention to the case of a static

social system. Actually, the properties of a social system itself

are expressed quite well, as a rule, in terms of the properties

of its behavors. So the present section may be looked upon

also as a treatment of selected properties of social systems in

the static case.

To clarify what particular properties are gained for behavors

from what properties of the behavior spaces, utility functions,

impression functions, interpretation functions and incentive

functions, the results of the present section are displayed in

as "disaggregated" form as is feasible here.

3.2.1   THEOREM: Let $X = X_\alpha \times X^\alpha$ be a compact (Hausdorff) space, let

f: $X \to R$ be continuous, and let $\bar{f}:X^\alpha \to R$ be defined by $\bar{f}(x^\alpha) = $

Sup $f(\cdot, x^\alpha)$. Define $\alpha:X^\alpha \to X$ by

$$\alpha(x^\alpha) = \{x_\alpha | f(x_\alpha, x^\alpha) \geq \bar{f}(x^\alpha)\}.$$

Then the graph $\Gamma(\alpha) \subset X$ of α is compact and, hence, α is upper

semi-continuous with $\alpha(x^\alpha)$ non-empty and compact $(x^\alpha \in X^\alpha)$.

Proof: From the continuity of f, it follows that $Y = f(X)$ is

compact and, by 3.1.4.3, that $\bar{f}$ is continuous. Since R is Hausdorff,

the graph $\Gamma(f) \subset X \times Y$ of f and the graph $\Gamma(\bar{f}) \subset X^\alpha \times Y$ of $\bar{f}$

are then both closed, hence compact. Thus, $X_\alpha \times \Gamma(\bar{f})$ is compact

and so is $(X_\alpha \times \Gamma(\bar{f})) \cap \Gamma(f)$. $= B$. Hence, the projection $\pi_X(B)$

is compact. Obviously, $\Gamma(\alpha) = \pi_X(B)$. For each $x^\alpha \in X^\alpha$, $\alpha(x^\alpha)$ is

non-empty by the compactness of $X_\alpha \times \{x^\alpha\}$ and the continuity of

f; it is compact, since it is the projection $\pi_{X_\alpha}((X_\alpha \times \{x^\alpha\}) \cap \Gamma(\alpha))$

of a compact set. Finally, α is upper semi-continuous by

closedness of $\Gamma(\alpha)$ and compactness of X_α , using Lemma 2 of

[Fan, 1952].

3.2.2 COROLLARY: If the collective behavior space of a static social

system is compact and $\ddot{w}_\alpha$ is continuous for (each) behavor α , then

(each) α has a compact graph and, hence is upper semi-continuous,

selecting a compact and non-empty choice set in its behavior

space in reaction to each α-exclusive behavior.

Proof: Replace f in 3.2.1 by $\ddot{w}_\alpha$.

#

3.2.3 COROLLARY: The consequence in 3.2.2 holds if the collective

behavior space is compact and u_α, h_α and $\ddot{g}_\alpha$ are continuous

Proof: If u_α, h_α and $\ddot{g}_\alpha$ are all continuous,

\#

3.2.4 <u>LEMMA</u>: Let Z be convex in a real vector space and let $g: Z \to R$ $u: Z \times R \to R$ be two real-valued functions, such that u is non-decreasing in $r \in R$, i.e., such that for all $z \in Z$, if r, $s \in R$, and $r \le s$, then $u(z,r) \le u(z,s)$. Define $f: Z \to R$ by $f(z) = u(z,g(z))$. Then f is quasi-concave if g is concave and u is quasi-concave.

Proof: If Z is empty, then there is nothing to prove. So let z, $z' \in Z$, $\lambda = 1 - \lambda' \in [o,1]$, and denote $\bar{z} = \lambda z + \lambda' z'$. Then

$$f(\bar{z}) = u(\bar{z}, g(\bar{z}))$$
$$\ge u(\bar{z}, \lambda g(z) + \lambda' g(z'))$$
$$\ge \text{Min } [u(z,g(z)), u(z',g(z'))]$$
$$= \text{Min } [f(z), f(z')].$$

\#

3.2.5 <u>COROLLARY</u>: If the behavior space X_α is convex in a real vector space and $Y^\alpha = h_\alpha(X^\alpha)$, then $\ddot{w}_\alpha$ is quasi-concave on $X_\alpha \times \{y^\alpha\}$ if $\ddot{g}_\alpha$ is concave on $X_\alpha \times \{y^\alpha\}$ and u_α is quasi-concave on $X_\alpha \times \{y^\alpha\} \times \ddot{g}_\alpha(X_\alpha \times \{y^\alpha\})$ $(v^\alpha = h_\alpha(x^\alpha), x^\alpha \in X^\alpha)$.

Proof: A direct application of 3.2.4.

\#

3.2.6 <u>COROLLARY</u>: If, in addition to the hypothesis of 3.2.5, X_α is compact and convex in a real linear topological space, and for each $y^\alpha \in Y^\alpha$, $\ddot{g}_\alpha$ and u_α are, respectively, continuous on $X_\alpha \times \{y^\alpha\}$

and upper semi-continuous on $X_\alpha \times \{y^\alpha\} \times \ddot{g}_\alpha$ ($X_\alpha \times \{y^\alpha\}$), then

$\alpha(x^\alpha)$ is non-empty, compact and convex for each $x^\alpha \in X^\alpha$.

Proof: Since X_α is compact, so is $X_\alpha \times \{y^\alpha\}$. Since $\ddot{g}_\alpha$ is

continuous so is $\ddot{g}_\alpha$ ($X_\alpha \times \{y^\alpha\}$) compact. By the same reason

and the upper semi-continuity of u_α, $\ddot{w}_\alpha$ is upper semi-continuous

on $X_\alpha \times x^\alpha$, for any $x^\alpha \in X^\alpha$. Thus, for each $x^\alpha \in X^\alpha$, $\ddot{w}_\alpha$ attains

a supremum $s^*(x^\alpha)$ on some $(x^*_\alpha, x^\alpha) \in X_\alpha \times \{x^\alpha\}$. Since $\ddot{w}_\alpha$ is

upper semi-continuous on $X_\alpha \times \{x^\alpha\}$, $\alpha (x^\alpha) = \{x_\alpha \in X_\alpha | \ddot{w}_\alpha (x_\alpha, x^\alpha)$

$\geq s^*(x^\alpha)\}$ is closed, hence compact, while obviously non-empty

from the previous sentence, $(x^\alpha \in X^\alpha)$. From 3.2.5, $\ddot{w}_\alpha$ is also

quasi-concave on $X_\alpha \times \{x^\alpha\}$, so that $\alpha(x^\alpha)$ is also convex $(x^\alpha \in X^\alpha)$.

#

3.3   Topological Properties of Behavors in Dynamic Social Systems

The present section is offered to serve the analysis of

dynamic social systems in a role analogous to that of the last

section for the case of static social systems. In a narrow sense,

the section is aimed at the existence theory of equilibrium for

dynamic social systems, presented in 4.3, but the facts recorded

are actually of wider interest.

As the last section spells out - albeit for the static case -

how various properties of its behavors derive from those of other

elements of a social system, the present section avoids the

corresponding exercise in the dynamic case. This is in the belief

that the last section provides an easy enough model to imitate.
The present section can, therefore, afford to be briefer.

3.3.1 THEOREM: Let X_α and X^α be compact (Hausdorff) spaces,
and let $\ddot{w}_\alpha: X_\alpha \times X^\alpha \to R$ be continuous. For generic $k_\alpha \in C(X_\alpha)$
and $x^\alpha \in X^\alpha$, denote $\bar{\ddot{w}}_\alpha = \underset{k_\alpha}{\text{Sup}}\ \ddot{w}_\alpha(\cdot, x^\alpha)$, and define

$$\alpha(k_\alpha, x^\alpha) = \{x_\alpha \in k_\alpha | \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \bar{\ddot{w}}_\alpha(k_\alpha, x^\alpha)\}$$

Then, taking the finite topology on $C(X_\alpha)$, the graph
$\Gamma(\alpha) \subset C(X_\alpha) \times X^\alpha \times X_\alpha$ is compact and, hence, α is usc,
with each $\alpha(k_\alpha x^\alpha)$ non-empty and compact.

Proof: By continuity of $\ddot{w}_\alpha$, the set

$$B = \{(k_\alpha, x^\alpha, x_\alpha, r) | r \leq \ddot{w}_\alpha(x_\alpha, x^\alpha)\}$$

is closed. By continuity of $\ddot{w}_\alpha$, $\bar{\ddot{w}}_\alpha$ is continuous, so that the
set

$$\Gamma_+(\bar{\ddot{w}}) = \{(k_\alpha, x^\alpha, r) | r \geq \ddot{w}(k_\alpha, x^\alpha)\}$$

is closed also. Hence $B \cap (\Gamma_+(\bar{\ddot{w}}) \times X_\alpha)$ is closed. In fact, it is
compact as a subset of the compact $C(X_\alpha) \times X^\alpha \times X_\alpha \times \bar{\ddot{w}}_\alpha(X_\alpha \times X^\alpha)$,
so that its projection P into $C(X_\alpha) \times X^\alpha \times X_\alpha$ is compact.
Now P is simply the set

$$P = \{(k_\alpha, x^\alpha, x_\alpha) | \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \bar{\ddot{w}}_\alpha(k_\alpha, x^\alpha)\}.$$

Defining the map $L':C(X_\alpha) \times X^\alpha \to X_\alpha$ by

$$L'(k_\alpha, x^\alpha) = \{x_\alpha | x_\alpha \in k_\alpha\},$$

its graph $\Gamma(L') \subset C(X_\alpha), \times X^\alpha \times X_\alpha$ is closed, so that $P \cap \Gamma(L')$ is compact. Observing that $\Gamma(\alpha) = P \cap \Gamma(L')$ completes the proof.

$\#$

3.3.2 <u>COROLLARY</u>: Let L_α and L^α be real linear T_1 spaces, let $X_\alpha \in kQ(L_\alpha)$ and let $X^\alpha \in kQ(L^\alpha)$. Let $\ddot{w}_\alpha : X_\alpha \times X^\alpha \to R$ be continuous and assume that, for each $x^\alpha \in X^\alpha$, $\ddot{w}_\alpha$ is quasi-concave on $X_\alpha \times \{x^\alpha\}$. Define $\alpha: CQ(X_\alpha) \times X^\alpha \to X_\alpha$, for generic $k_\alpha \in CQ(X_\alpha)$ and $x^\alpha \in X^\alpha$, by

$$\alpha(k_\alpha, x^\alpha) = \{x_\alpha \in k_\alpha | \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \bar{\ddot{w}}_\alpha(k_\alpha, x^\alpha)\},$$

where $\bar{\ddot{w}}_\alpha(k_\alpha, x^\alpha) = \underset{k_\alpha}{Sup}\ \ddot{w}_\alpha(\cdot, x^\alpha)$. Then α is usc with compact graph and with each $\alpha(k_\alpha, x^\alpha)$ non-empty, compact and convex, taking the finite topology on $CQ(X_\alpha)$.

<u>Proof</u>: All but the fact that each $\alpha(k_\alpha, x^\alpha)$ is convex follows from the last theorem, for a linear T_1 space is certainly T_2 (Hausdorff). Given any $(k_\alpha, x^\alpha) \in CQ(X_\alpha) \times X^\alpha$, the quasi-concavity of $\ddot{w}_\alpha$ on $X_\alpha \times \{x^\alpha\}$ ensures the convexity of the set

$$\{x_\alpha \in X_\alpha | \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \bar{\ddot{w}}_\alpha(k_\alpha, x^\alpha)\}.$$

But $\alpha(k_\alpha, x^\alpha)$ is nothing but the intersection of this set with the convex k_α, so $\alpha(k_\alpha, x^\alpha)$ is also convex.

$\#$

4. EVOLUTION AND EQUILIBRIUM IN SOCIAL SYSTEMS

This chapter is concerned with equilibrium in social systems. The notion of 'equilibrium' is briefly compared with those of 'Pareto point' and 'core point' as treated in economic theory and the theory of games. Equilibrium points are then considered as fixed points of certain mappings, called "evolutions", representing the way in which certain adjustment processes operate for various types of social system.

Of the five very general types of social system studied for the existence of an equilibrium, the first (type 0) is static. The remaining types (I-IV) are dynamic. All five types, classified according to various topological properties which they satisfy, are unrestricted in the cardinality of their personnel, so that the personnel can be finite, countably infinite or uncountably infinite. The behavior spaces are compact and convex in locally convex real linear topological spaces in the case of type 0 social systems; they are compact and convex in normed real linear topological spaces in the case of types I-IV. For all five types of social system, the set of equilibria is shown to be non-vacuous, compact, and, in certain cases, also convex.

The existence result for equilibrium in the case of type 0 social systems is proved by use of (effectively a fixed point) theorem of Fan (1952). In social systems of types I-IV, the existence of social equilibirum is established by applying a fixed point theorem of Prakash and Sertel (1970 a) for certain (real semi-

linear topological) spaces developed by these authors (1970 b).

These latter existence results are extensions - obtained by

encorporation of impression, incentive and interpretation

schemes - of results developed by Prakash and Sertell (1970 c).

4.1 Notions of Social Equilibrium

Of the many different notions of 'equilibrium' that are
relevant to the study of social systems, those which have attracted
the most attention in the study of games and economies are three
kinds. Only one of these really goes under the name of "equilibrium",
although the other two "Pareto point" and "core point", are also
equilibria in a real sense and sometimes recognized as such. While
the present study is concerned mainly with the first of these and
it is the existence of equilibrium in this particular sense that
is established in the other sections of this chapter for certain
types of social systems, the study might gain perspective if the
relation between the mentioned three kinds of equilibrium is
indicated. (To adhere to the most common usage, "equilibrium"
will hereafter be used to refer to the first mentioned of these.)

To define 'equilibrium', 'Pareto point' and 'core point' in
a fashion that will allow easy comparison, a few preparatory
definitions are in order. These become simpler in the case of
a game or, in general, a static social system, since the notion
of 'admissibility' loses its importance in that case, as the
reader will notice for himself immediately the definition is
given. The notion of 'blocking' by "coalitions" is pivotal for
the comparison between the three kinds of point (equilibrium,
Pareto and core).

4.1.1. <u>Definition</u>: Let A be the personnel of a social system S. Any

non-empty subset $B \subset A$ will be called a <u>coalition</u> of S, and

the notation $X_B = \prod_B X_\alpha$, $X^B = \prod_{A \setminus B} X_\alpha$, $D_B = \prod_B D_\alpha$, $D^B = \prod_{A \setminus B} D_\alpha$, will

be adopted with generic elements denoted as $x_B \in X_B$, $x^B \in X^B$,

$d_B \in D_B$, $d^B \in D^B$.

4.1.2. <u>Definition</u>: Let x be a collective behavior and d a collective

feasibility of a social system S with personnel A. The pair

(x,d) will be said to be <u>inadmissible for</u> α ($\alpha \in A$) iff

$\pi_{X_\alpha}(x) = x_\alpha \notin d_\alpha = \pi_{D_\alpha}(d)$; otherwise, it will be said to

be <u>admissible for</u> α; (x,d) will be said to be <u>admissible for B</u>,

a coalition of S, iff (x,d) is admissible for each $\alpha \in B$;

otherwise, it will be said to be <u>inadmissible for B.</u> (x,d)

will be said to be <u>(in)admissible</u> iff it is (in)admissible for

A.

4.1.3 <u>Definition</u>: Let S be a social system, Let B be a coalition

of S, and let $x = (x_B, x^B)$, $d = (d_B, d^B)$ be a collective

behavior and a collective feasibility, respectively. If

the pair (x,d) is admissible for B, then it will be said

to be <u>blocked by B</u> iff there exists $y_B \in d_B$ such that

$$\ddot{w}_\beta(y_B, x^B) \geq \ddot{w}_\beta (x_B, x^B)$$

holds for all $\beta \in B$ and

$$\ddot{w}_{\beta_*} (y_B, x^B) > \ddot{w}_{\beta_*} (x_B, x^B)$$

holds for some $\beta \in B$. If (x,d) is admissible for B, then it will

be said to be <u>unblocked by B</u> iff it is not blocked by B.

The notions of 'equilibrium', 'Pareto point' and 'core point'

can now be given precise meaning by the following defintion.

4.1.4   <u>Definition</u>: Let x be a collective behavior and d a collective

feasibility of a social system S with personnel A, such that

the pair (x,d) is admissible. The pair (x,d) will be called

an <u>equilibrium</u> (point) of S iff it is unblocked by each singleton

coalition {α} A; it will be called a <u>Pareto point</u> of S iff it

is unblocked by A; it will be called a <u>core point</u> of S iff it

is unblocked by all coalitions of S. The <u>equilibrium set</u> of

S is the set of all equilibria of S. The <u>Pareto set</u> of S

is the set of all Pareto points of S; the <u>core</u> of S is the set

of all core points of S.

The comparison between equilibrium, Pareto and core points of

a social system is now absolutely clear. Simple but important

consequences of the definitions are that all core points are

equilibria as well as Pareto points, so that the core is contained in

the intersection of the equilibrium set with the Pareto set.

Thus, the core is empty when, for instance, S does not have

an equilibrium. In the later

sections of this chapter, sufficient conditions are given

for S to have an equilibrium. Demonstrating sufficient

conditions for the Pareto set or the core to be

non-vacuous is a research problem not tackled in this study.

Although it certainly will not be a survey of the

literature, a brief sketch of the history of the above ideas

will now be given. The idea of an equilibrium point for a

competitive economy, in the sense of a price vector equating

supply to demand in each market, is commonly attributed

to Walras (1881). Wald (1933-4, 1934-5, 1951) proved,

under rather restricted conditions, that an equilibrium

exists for each of a pure production and a pure exchange

economy. Arrow and Debreu (1954), based on a study of Debreu

(1952), demonstrated for the first time the existence of an

equilibrium for a competitive economy in which production,

exchange and consumption all take place, using less

stringent assumptions than those of Wald. One very good

reason why a survey of the literature up to 1954 is not

given here is that the mentioned work of Arrow and Debreu

contains an excellent such survey. The existence theorem

of Debreu (1952) for a social equilibrium (where, however,

the "social system" is actually the same as that defined to

be an "economy" in Arrow and Debreu (1954) - see also section

2.2 of the present study) is used in establishing the result

of Arrow and Debreu (1954), and that theorem is itself based on a corollary of the fixed point theorem of Eilenberg and Montgomery (1946). Using Kakutani's (1941) fixed point theorem, McKenzie (1955) obtained improvements on the Arrow and Debreu study. All of these works used finite dimensional methods. They succeeded in providing economics with the equilibrium whose various optimality and stability aspects had long been discussed and many worked out by a long and formidable list of authors. Newman's (1968) excellent collection is one good entry into the box of gems to which all of the above belong.

The idea of an equilibrium for a game was first formally introduced in some generality by Nash (1950) and proved to exist by him (1950, 1951), first by use of Kakutani's and then by use of Brouwer's fixed point theorem. The game dealt with was a finite personnel non-cooperative one with behavior spaces (compact and convex) in Euclidean spaces.

For information concerning the Pareto set, the core and the relations between these and the equilibrium set, the following short list might provide a reasonable means of entry into the associated game-theoretic and economic literature: (Edgeworth, 1881), (von Neumann and Morgenstern, 1944), (Arrow, 1950) ,(Gillies, 1953, 1959), (Bondareva,1962) (Debreu and Scarf, 1963); (Vind, 1964), (Aumann, 1964), (Shapley, 1965), Scarf, 1967). Of these (Aumann, 1964)

contains a short but illuminating discussion of the development

of the area.

4.2   Evolution and Equilibrium in Static Social Systems

In this section an equilibrium of a static social system is
defined as a fixed point of a certain transformation called an
"evolution", and a sufficient condition is demonstrated for the
existence of such an equilibrium. The "static contractual set",
i.e., the set of equilibria of a static social system, is shown
to be compact if the mentioned sufficient condition is satisfied.
This contractual set is shown also to be convex if certain
linearity conditions hold which yield the effective utility
function^s linear.

4.2.1   Notation and Convention: In the case of static social systems
it is possible to somewhat simplify the notation adopted for
dealing with social systems in general. This simplification is
permitted by the fact that the feasibility transformations t_α of
a static social system are all constant functions, the typical
t_α assigning a fixed $d_\alpha \subset X_\alpha$ to every point in its domain.
From this fact it is clear that no generality is lost by
assuming $d_\alpha \cdot = X_\alpha$, i.e., $D_\alpha = \{X_\alpha\}$, for all the behavors
$\alpha \in A$. Taking advantage of this, the set $T \in S$ can be fully
specified and suppressed by representing a static social system
S in the form $\langle W, U, H, G, I A \rangle$. For, whenever a social
system is specified in this way, it will be understood to be
static and the constant collective feasibility will be taken to be

the collective behavior space X. In this case, since there

is always one constant feasibility d_α = X_α for each behavor

$\alpha \epsilon$ A, the behavors become simply point-to-set mappings

$\alpha: X^\alpha \to X_\alpha$ defined by

$$\alpha(x^\alpha) = \{x_\alpha \epsilon X_\alpha | \ddot{w}_\alpha(x_\alpha, x^\alpha) \geq \ddot{\bar{w}}_\alpha(x^\alpha)\} \ ,$$

where

$$\ddot{\bar{w}}_\alpha(x^\alpha) = \underset{y_\alpha \epsilon X_\alpha}{\text{Sup}} \ \ddot{w}_\alpha(y_\alpha, x^\alpha).$$

4.2.2 <u>Definitions</u>: The <u>evolution of a(static)</u> social system

S = <W, U, H, G, I, A> is a transformation E: X → [X] defined

for each x ϵ X, by

$$E(x) = \underset{A}{\Pi} \ \alpha(\pi_{X\alpha}(x)).$$

4.2.3 <u>Definition</u>: The <u>(static) contractual set</u> of a static social

system S is the set $C = \{x \epsilon X | x \epsilon E(x)\} \subset X$ of fixed points

of the evolution E of S. A collective behavior x ϵ X is called

a <u>(static)social contract</u> or <u>equilibrium</u> of S iff x ϵ C.

4.2.4. <u>Definition</u>: A static social system S will be classified as

<u>type 0</u> ("type zero") iff the following conditions are satisfied

for each behavor $\alpha \epsilon$ A:

(1) $X_\alpha \in KQ(L_\alpha)$ for some locally convex real linear

topological space L_α :

(2) $\ddot{w}_\alpha$ is continuous on X and quasi-concave on $X_\alpha \times \{x^\alpha\}$ for each

$x^\alpha \in X^\alpha$.

The main theorem, 4.2.6, of this section will be proved

by use of the following (fixed point) theorem.

4.2.5 THEOREM [Fan, 1952]: Let $\{L_\alpha | \alpha \in A\}$ be a family of locally

convex real linear topological spaces. For each $\alpha \in A$, let

$X_\alpha \in KQ(L_\alpha)$ and let $X^\alpha = \underset{A\backslash\{\alpha\}}{\Pi} X_\beta$. Let $X = \underset{A}{\Pi} X_\alpha$ and let

$\{\Gamma(\alpha) | \alpha \in A\}$ be a family of closed subsets of X. If,

for any point $x \in X$, and for any $\alpha \in A$, the set $\alpha(x^\alpha) \in O(X_\alpha)$,

where $x^\alpha = \pi_{X\alpha}(x)$ and

$$\alpha(x^\alpha) = \pi_{X_\alpha}(\Gamma(\alpha) \cap (X_\alpha \times \{x^\alpha\})),$$

then $\underset{A}{\cap}\Gamma(\alpha) \neq \phi$.

4.2.6 THEOREM: Every type 0 social system has an equilibrium.

Proof: By 3.2.2, the graph $\Gamma(\alpha)$ of each behavor α in

the personnel A satisfies all but the requirement that

$\alpha(x^\alpha) \subset X_\alpha$ is convex for each $x^\alpha \in X^\alpha$. This requirement

is satisfied, however, since $\ddot{w}_\alpha$ is quasi-concave. Thus,

$\bigcap_{A} \Gamma(\alpha) \neq \phi$ and, evidently any point x in this intersection is
a fixed point of the evolution, i.e., an equilibrium.

#

4.2.7   COROLLARY: A static social system with a collective behavior

space which is compact and convex in a locally convex real

linear topological space is of type 0, hence has an equilibrium,

if the following conditions are satisfied for each behavor

α in the personnel:

   (1) u_α is quasi-concave on $X_\alpha \times \{y^\alpha\}$ for each $y^\alpha \in h_\alpha(X^\alpha)$,

         and u_α is continuous;

   (2) h_α is continuous;

   (3) $\ddot{g}_\alpha$ is concave on $X_\alpha \times \{y^\alpha\}$ for each $y^\alpha \in h_\alpha (X^\alpha)$ and

         $\ddot{g}_\alpha$ is continuous.

Proof: From the quasi-concavity and concavity of u_α, $\ddot{g}_\alpha$, respec-

tively, on $X_\alpha \times \{y^\alpha\}$, it follows by 3.2.4   that $\ddot{w}_\alpha$ is quasi-

concave on the same $(y^\alpha \in h_\alpha(X^\alpha)$; $\alpha \in A)$. Hence $\alpha(x^\alpha)$

is convex for each $x^\alpha \in X^\alpha$ and $\alpha \in A$.   From the continuity of

u_α , h_α and $\ddot{g}_\alpha$, it follows that $\ddot{w}_\alpha$ is continuous, so that the

social system is of type 0.   Now 4.2.6 applies.

#

4.2.8   THEOREM: The contractual set C of a type 0 social system is

compact.

Proof: Clearly, C is precisely the intersection of the graphs $\Gamma(\alpha)$ of all the behavors, which is compact since each $\Gamma(\alpha)$ is compact by 3.2.2.

#

4.2.9 THEOREM: The contractual set of a type 0 social system S is convex if the effective utility function $\ddot{w}_\alpha$ of each behavor α in the personnel of S is linear (both concave and convex).

Proof: It suffices to show that the graph $\Gamma(\alpha)$ of an arbitrary behavor α in the personnel is convex. Let α be such a behavor, and let x, y, ε $\Gamma(\alpha)$, where x = (x^α, x_α) and y = (y^α, y_α). Define z = $(z^\alpha, z_\alpha) = \lambda x + \lambda' y$ for any $\lambda = 1 - \lambda'$ $\varepsilon[0,1]$. Then $\ddot{w}_\alpha(z) = \lambda\ddot{w}_\alpha(x) + \lambda' \ddot{w}_\alpha(y)$, by linearity of $\ddot{w}_\alpha$. Since x, y $\varepsilon\Gamma(\alpha)$, $\ddot{w}_\alpha(x) = \bar{\ddot{w}}_\alpha(x^\alpha)$ and $\ddot{w}_\alpha(y) = \bar{\ddot{w}}_\alpha(y^\alpha)$. But, by 3.1.4.6.1.c and the linearity of $\ddot{w}_\alpha$, $\bar{\ddot{w}}_\alpha$ is linear also. Hence, $\bar{\ddot{w}}_\alpha(z^\alpha) = \lambda\bar{\ddot{w}}_\alpha(x^\alpha) + \lambda' \bar{\ddot{w}}_\alpha(y^\alpha)$, so that $\bar{\ddot{w}}_\alpha(z^\alpha) = \ddot{w}_\alpha(z)$, implying that $z_\alpha \varepsilon \alpha(z^\alpha)$, i.e., that z $\varepsilon\Gamma(\alpha)$. Thus, $\Gamma(\alpha)$ is convex, completing the proof.

4.2.10 COROLLARY: Let S be as in 4.2.9, and let A be the personnel of S. Denote $Y^\alpha = h_\alpha(X^\alpha)$ for each α ε A. Assume that, for each α ε A, $h_\alpha(\lambda x^\alpha + {}'\lambda{}'x^\alpha) = \lambda h_\alpha(x^\alpha) + {}'\lambda h_\alpha({}' x^\alpha)$ if x^α and ${}'x^\alpha$belong to X^α. Assume that $\ddot{g}_\alpha$ is linear (concave and

convex on $X_\alpha \times Y^\alpha$, for each $\alpha \in A$. If u_α is linear on

$X_\alpha \times Y^\alpha \times \ddot{g}_\alpha (X_\alpha \times Y^\alpha)$ for each $\alpha \in A$, then the contractual

set of S is convex.

Proof: It follows from the hypothesis concerning h_α and $\ddot{g}_\alpha$

that $\ddot{g}_\alpha (x^\alpha, h_\alpha(x^\alpha))$ is linear on X. It is easy to see

(though tedious to show) that the hypothesis concerning u_α then

ensures the linearity of $\ddot{w}_\alpha$. Since $\ddot{w}_\alpha$ is thus linear from

each $\alpha \in A$, the desired result follows by 4.2.9.

#

4.3. Evolution and Equilibrium in Dynamic Social Systems

The primary aim of this section is to demonstrate reasonably
unrestrictive sufficient conditions for the existence of what will
be defined to be a dynamic social equilibrium. Although only
one type of evolution and social equilibrium needed consideration
in the case of static social systems, in the case of dynamic
social systems several types of evolution and several corresponding
types of equilibrium deserve attention. For, in the dynamic case,
the feasibility transformations are no longer restricted to be
constant maps, so that various forms of relaxation of the constant-
ness of these maps can be considered in conjunction with various
assumptions governing the behavors, yielding a variety of
conditions each of which affords an existence theorem for an
associated type of social equilibrium. Furthermore, none of these
sufficient conditions implies the other, so the existence
theory for social equilibrium in dynamic social systems does not
reduce to a single theorem as it did in the static case - at least,
this author is not able to assert such a reduction at this time.

To compactify the statement of the existence theorems for
dynamic social equilibri , it is convenient earlier to have defined
types of social systems accordingly as they satisfy certain
conditions. In this way the sufficient conditions for existence of
an equilibrium are collected under each "type", and the hypotheses
of the existence theorems are shortened to become assumptions
that the social system is of one type or another. Also, collecting

all of these types in one subsection is intended to facilitate their comparison with each other. The following section is thusly motivated.

4.3.1 Types of Dynamic Social System

It helps the exposition to collect at the outset all the common features of the dynamic social systems to be considered here. The first common feature of these is that the behavior space X_α of each behavior α is assumed to be non-empty compact and convex in some normed real linear topological space L_α:

$$(4.3.1.1) \qquad X_\alpha \in kQ(L_\alpha) \qquad (\alpha \in A).$$

Secondly, the effective utility function, $\ddot{w}_\alpha$, of each behavor $\alpha \in A$ is assumed to be continuous (on X) and (its restriction) quasi-concave on $X_\alpha \times \{x^\alpha\}$ for each $x^\alpha \in X^\alpha$:

(i) $\ddot{w}_\alpha : X \to R$ is continuous,

$$(4.3.1.2) \qquad\qquad\qquad\qquad\qquad\qquad (\alpha \in A).$$

(ii) $\ddot{w}_\alpha : X_\alpha \times \{x^\alpha\} \to R$ is quasi-concave

$$(x^\alpha \in X^\alpha)$$

Finally, the feasibility space D_α of each behavor $\alpha \in A$ is assumed to be the Hausdorff metric space (see 4.3.1.5) of all

non-empty closed (hence compact) and convex subsets of X_α:

$$(4.3.1.3) \qquad D_\alpha = CQ(X_\alpha) \qquad (\alpha \in A).$$

This gives simplicity to the following definition.

4.3.1.4   Definition: A social system satisfying 4.3.1.1-3 is classified

as type I iff the feasibility transformation

$$t_\alpha: \prod_A CQ(X_\alpha) \times X \to CQ(X_\alpha)$$

is continuous for each behavor $\alpha \in A$

4.3.1.5   LEMMA: If X is compact and convex in a metric linear space

and C(X) has the Hausdorff metric [or, equivalently (see 3.1.2.13),

the finite] topology, then C(X) and its subspace CQ(X) are both

compact and convex.

Proof: The compactness of C(X) follows by 3.1.2.12.   The

convexity of C(X) follows by the continuity of scalar multi-

plication and vector addition in the linear topological space

where X lies.   Since convex combinations of convex sets are

convex, it follows also that $CQ(X) \subset C(X)$ is convex, and the

following simple convergence argument establishes what is

needed.   Let $\{^i k\}_{i=1}^\infty \to {}^* k$ be a converging sequence of points

$^i k \in CQ(X)$.   Then $^* k \in C(X)$, since C(X) is closed.   Let $^* x$,

$^* x' \in {}^* k$, and denote an arbitrary convex combination

$\lambda^{*}x + \lambda'^{*}x'$ by $^{*}\bar{x}$ $(\lambda = 1 - \lambda' \in [0,1])$. Then there exist

sequences $\{^{i}x\}_{i=1}^{\infty} \to {}^{*}x, \{^{i}x'\}_{i=1}^{\infty} \to {}^{*}x'$ with ${}^{i}x, {}^{i}x' \in {}^{i}k$

$(i = 1,2, \ldots)$. Since ${}^{i}k$ is convex, $\lambda {}^{i}x + \lambda' {}^{i}x' = {}^{i}\bar{x} \in {}^{i}k$

$(i = 1, 2, \ldots)$, so that $\{^{i}\bar{x}\}_{i=1}^{\infty} \to {}^{*}\bar{x}$. Hence, $^{*}\bar{x} \in {}^{*}k$, proving

that $^{*}k$ is convex and, thus, that $CQ(X)$ is closed.

$\#$

In the next two definitions one is able to relax the

condition on the feasibility transformations by restricting the

effective utility functions in alternate ways.

4.3.1.6 Definition: A social system satisfying 4.3.1.1-3 is classified

as type II iff, for each behavior $\alpha \in A$, the effective utility

function $\ddot{w}_{\alpha}$ is "linear" (both concave and convex) and the

feasibility transformation t_{α} is usc as a point-to-set mapping

with t_{α} $(k, x) \subset CQ(X_{\alpha})$ being closed and convex,

$(k \in \prod_{A} CQ(X_{\alpha}), \quad x \in X)$.

4.3.1.7 Remark: In this case, it is possible to view t_{α} as a

point-to-point mapping into $CQ(CQ[X_{\alpha}])$, $(\alpha \in A)$.

4.3.1.8 Definition: A social system satisfying 4.3.1.1-3 is classified

as type III iff, for each behavor $\alpha \in A$, the effective

utility function $\ddot{w}_{\alpha}$ is strictly quasi-concave on each $X_{\alpha} \times \{x^{\alpha}\}$

$(x^{\alpha} \in X^{\alpha})$ and the feasibility transformation t_{α} is as in type II.

Finally, these last two restrictions on the effective utility functions can be eliminated with appeal to an interesting restriction on the feasibility transformations.

4.3.1.9   Definition: A mapping $\xi: X \to Y$ of a convex set X into a convex set Y is called a convex process iff its graph $\Gamma(\xi) \subseteq X \times Y$ is convex.

4.3.1.10   Definition: A social system satisfying 4.3.1.1-3 is classified as type IV iff the collective feasibility transformation is an usc convex process.

4.3.1.11   Remark: Of course, a collective feasibility transformation t is a convex process iff each coordinate feasibility transformation is a convex process.

This completes the present classification of social systems into types. It is clear that, if the feasibility transformations are all constant maps, a social system of type I - IV is slightly more restricted than that dealt with in section 4.2. The restriction comes from the fact that each X_α is now assumed to lie in a normed (real linear topological) space, while before it was only assumed to lie in a locally convex (real linear topological) space. Thus, by taking on this restriction, the constancy assumption for the feasibility transformations has been

eliminated.

Before turning in section 4.3.3 to prove the existence of equilibrium for all four types of social system defined, the next brief section will identify types of evolutions and equilibria corresponding to these types of social system.

4.3.2 Types of Evolution and Equilibrium for Dynamic Social Systems

Corresponding to the types of social system   defined in 4.3.1, a set of mappings will now be defined. Corresponding to each such mapping, called an "evolution", a type of social equilibrium will be identified as a fixed point of that mapping. It is such types of fixed point which will be shown in the next section to exist.

4.3.2.1 Notation: Let k, ℓ denote generic elements of $\prod_{A} CQ(X_\alpha)$ and let x, y denote generic elements of X. Denote $\pi_{CQ(X_\alpha)}(k) = k_\alpha$, $\pi_{CQ(X_\alpha)}(\ell) = \ell_\alpha$, $\pi_{X_\alpha}(x) = x_\alpha$, $\pi_{X\alpha}(x) = x^\alpha$, $\pi_{X_\alpha}(y) = y_\alpha$. Also denote $Z_\alpha = CQ(X_\alpha) \times X_\alpha$, $Z = \prod_{A} Z_\alpha$, $\underline{A}(k,x) = \prod_{A} \alpha(k_\alpha, x^\alpha)$. Finally, denote generic elements of Z also by z.

4.3.2.2 Definition: The evolution of a type I social system is a transformation $E: Z \to Z$ defined for $z \in Z$ by

$$E_I(z) = \{t(z)\} \times \underline{A}(t(z), x);$$

it is classified as type I. The evolution of a type II social system is a transformation $E_{II}: Z \to Z$ defined, for

$z \in Z$, by

$$E_{II}(z) = t(z) \times \underline{A}(t(z),z);$$

it is classified as type II. The evolutions of a type

III social system are transformations $E_{III}:Z \to Z$ and

$E_{IV}: Z \to Z$ defined, for $z = (k,x) \in Z$ by

$$E_{III}(k,x) = (\underset{A}{\Pi} \; t_{\alpha}(k, \; \alpha(k_{\alpha},x^{\alpha}),x^{\alpha})) \times \underline{A}(k,x),$$

$$E_{IV}(k,x) = t(\{k\} \times \underline{A}(k,x)) \times \underline{A}(k,x),$$

respectively; E_{III} is classified as type III and E_{IV} as

type IV . The evolution of a type IV social system is a

transformation $E_{IV}:Z \to Z$ defined and classified as above.

4.3.2.3   Definition: A fixed point of an evolution is called a social

equilibrium (or social contract) and classified according to the

type of evolution of which it is a fixed point.

So much preparation finally allows turning to the existence

of social equilibrium in the dynamic case.

4.3.3   Existence of Dynamic Social Equilibrium

In this section it is shown that social systems of types I - IV have equilibria. In particular, it is shown that the (type i) evolution of a type i social system has a (type i) social contract for $i \in \{I, II, III, IV\}$, and that a type III social system has, in addition, a type IV social contract.

In each of the existence results, the following fixed point theorem is used:

4.3.3.1   <u>THEOREM [Prakash and Sertel]</u>: Let $\{Z_\alpha | \alpha \in A\}$ be a family of 2° convex, compact and convex spaces, and let $\{E_\alpha :$ $Z \to Z_\alpha \mid \alpha \in A\}$ be a family of usc point-to-set mappings on $Z = \prod_A Z_\alpha$ such that $E_\alpha(z) \in CQ(Z_\alpha)$ $(z \in Z, \alpha \in A)$. Define $E: Z \to Z$ by $E(z) = \prod_A E_\alpha(z)$ $(z \in Z)$. Then there exists a (fixed)point $z \in Z$ such that $z \in E(z)$. [1970, 3.16 Theorem VI].

In each application of this theorem, Z_α will be taken to be $CQ(X_\alpha) \times X_\alpha$, as indicated in 4.3.2.1. In this case, both $CQ(X_\alpha)$ and X_α are easily seen to be 3° convex, so that their product Z_α is 3° convex, hence 2° convex.

It is worth noting that the fixed point theorem of Fan, used in 4.2, cannot be applied in this case, as $Z_\alpha = CQ(X_\alpha) \times X_\alpha$ lies in a <u>semi</u>-linear topological space while it must be assumed to lie in a linear topological space for Fan's mentioned theorem to apply.

Thus, an extension of Fan's theorem is needed, and 4.3.3.1

provides just the desired sort of extension. Actually, other

fixed point theorems of Prakash and Sertel [1970] can safely

be conjectured to suffice for some of the above types of

social system, but 4.3.3.1 works for all of these, as will be

seen. It is because 4.3.3.1 works for all of these cases that it is

possible to economize on the number of fixed point theorems to be

used (known).

4.3.3.2   THEOREM: Every type I social system has a type I social contract.

Proof: By 3.3.2, each behavor $\alpha \in A$ is usc in $(t_\alpha(k,x),x^\alpha)$,

with $\alpha(t_\alpha(k,x), x^\alpha) \in KQ(X_\alpha) = CQ(X_\alpha)$ for each $(k,x) \in Z$.

Since each t_α and $\pi_{X^\alpha}(k,x) = x^\alpha$ is continuous

on Z, $\alpha(t_\alpha(k,x),x^\alpha)$ is usc on Z. Hence, $\{t_\alpha(k,x)\} \times \alpha(t_\alpha(k,x),x^\alpha)$

is usc on Z, while $\{t_\alpha(k,x)\} \times \alpha(t_\alpha(k,x), x^\alpha) \in CQ(Z_\alpha)$

$[(k,x) \in Z, \alpha \in A]$. Also, $Z = \Pi_A Z_\alpha$ is the product of 2^O convex

compact and convex spaces Z_α, and

$$E_I(k,x) = \Pi_A (\{t_\alpha(k,x)\} \times \alpha(t_\alpha(k,x),x^\alpha))$$

Thus, by 4.3.3.1, there exists a type I social contract

$z \in E_I(z) \subset Z$.

#

4.3.3.3   THEOREM: Every type II social system has a type II social contract.

Proof: Since each t_α is usc, each $\alpha(t_\alpha(k,x), x^\alpha)$, as a composition of usc mappings, is usc on Z. Thus, $t(k,x) \times \alpha(t_\alpha(k,x), x^\alpha)$, for each $\alpha \in A$, is usc on Z. Since $F_{II}(k,x) = \prod_A E_{II}$, where

$$E_{II\alpha}(k,x) = t_\alpha(k,x) \times \alpha(t_\alpha(k,x),x^\alpha),$$

to apply 4.3.3.1 it remains only to show that $E_{II}(k,x) \in CQ(Z)$ for each $\alpha \in A$. Since $t_\alpha(k,x) \in CQ(CQ(X_\alpha))$ by hypothesis, it suffices to show that $\alpha(t_\alpha(k,x), x^\alpha) \in CQ(X_\alpha)$. In fact, as the graph $\Gamma(\alpha) \subset CQ(X_\alpha) \times X^\alpha \times X_\alpha$ is compact by 3.3.1, and $t_\alpha(k,x)$ is closed in the compact $CQ(X_\alpha)$, it follows that $t_\alpha(k,x) \times \{x^\alpha\} \subset CQ(X_\alpha) \times X^\alpha$ is compact, implying that $\alpha(t_\alpha(k,x), x^\alpha)$ is compact, hence closed. This leaves only the convexity of $\alpha(t_\alpha(k,x), x^\alpha)$ to prove. Let x_α, $x'_\alpha \in \alpha(t_\alpha(k,x), x^\alpha)$. Then $\ddot{w}_\alpha(x_\alpha,x^\alpha) = \ddot{\bar{w}}_\alpha(\ell_\alpha, x^\alpha)$ and $\ddot{w}_\alpha(x'_\alpha, x^\alpha) = \ddot{\bar{w}}_\alpha(\ell'_\alpha, x^\alpha)$, for some ℓ_α, $\ell'_\alpha \in t_\alpha(k,x)$. By 3.1.4.6, linearity of $\ddot{w}_\alpha$ implies that of $\ddot{\bar{w}}_\alpha$, so that $\ddot{\bar{w}}_\alpha(\bar{\ell}_\alpha, x^\alpha) = \lambda\ddot{\bar{w}}(\ell_\alpha,x^\alpha) + \lambda'\ddot{\bar{w}}(\ell'_\alpha,x^\alpha)$ for all $\lambda = 1 - \lambda' \in [0,1]$, where $\bar{\ell}_\alpha = \lambda\ell_\alpha + \lambda'\ell'_\alpha$. But, by linearity of $\ddot{w}_\alpha$, $\ddot{w}_\alpha(\bar{x}_\alpha, x^\alpha) = \lambda\ddot{w}_\alpha(x_\alpha, x^\alpha) + \lambda' \ddot{w}_\alpha(x'_\alpha,x^\alpha) = \ddot{\bar{w}}(\bar{\ell}_\alpha, x^\alpha)$, for $\bar{x}_\alpha = \lambda x_\alpha + \lambda'x'_\alpha$. Hence, $\bar{x}_\alpha \in \alpha(t_\alpha(k,x), x^\alpha)$, showing all that was required.

4.3.3.4 <u>THEOREM</u>: Every type III social system has (1) a type III

and (2) a type IV social contract.

<u>Proof</u>:

<u>ad(1)</u>: Since the effective utility function $\ddot{w}_\alpha$ is strictly

concave on $X_\alpha \times \{x^\alpha\}$ for each $x^\alpha \in X^\alpha$, $\alpha(k_\alpha, x^\alpha)$ is singleton

for each $k_\alpha \in CQ(X_\alpha)$, $x^\alpha \in X^\alpha$, $\alpha \in A$. Then $t_\alpha(k, \alpha(k_\alpha, x^\alpha), x^\alpha)$

is the image under t_α of a point $(k, \alpha(k_\alpha, x^\alpha), x^\alpha) \in Z$, hence it

is non-empty, compact and convex in $CQ(X_\alpha)$, by hypothesis, for

each such point $z \in Z(\alpha \in A)$. Thus, for each $z = (k,x) \in Z$,

$$E_{III_\alpha}(z) \in CQ(CQ(X_\alpha)) \times \{\{x_\alpha\}|x_\alpha \in X_\alpha\} \subset CQ(CQ(X_\alpha) \times \{\{x_\alpha\}|x_\alpha \in$$

$X_\alpha\})$, which is homeomorphic to $CQ(Z_\alpha)$, where, for each $\alpha \in A$,

$$E_{III_\alpha}(z) = t_\alpha(k, \alpha(k_\alpha, x^\alpha), x^\alpha) \times \alpha(k_\alpha, x^\alpha).$$

But the map $E_{III_\alpha}: Z \to CQ(Z_\alpha)$ is usc, as t_α is so and α and

all projections are continuous $(\alpha \in A)$, and $E_{III}(z) = \underset{A}{\Pi}E_{III_\alpha}(z)$

for any $z \in Z$. Hence, 4.3.3.1 applies, yielding that there

exists a type III social contract $z \in Z$ such that $z \in E_{III}(z)$.

<u>ad(2)</u>: Defining

$$E_{IV_\alpha}(k,x) = t_\alpha(\{k\} \times \underline{A}(k,x)) \times \alpha(k_\alpha, x^\alpha)$$

for each $\alpha \in A$ and $z = (k,x) \in Z$, $E_{IV}(z) = \underset{A}{\Pi}E_{IV_\alpha}(z)$. Since

each α is usc, so is the map $\underline{A}$. As each $\alpha(k_\alpha, x^\alpha)$ is

singleton, so is $\underline{A}(z)$ $(z \in Z)$. Hence, $\{k\} \times \underline{A}(k,x) \in Z$

for each $(k,x) \in Z$. Thus, $E_{IV_\alpha}: Z \to CQ(CQ(X_\alpha)) \times X_\alpha$ has been

defined for each $\alpha \in A$, and the range is a subspace of

$CQ(Z_\alpha^{\cdot})$, so that E_{IV} is the product of usc maps $E_{IV} : Z \to CQ(Z_\alpha)$.

By 4.3.3.1, there exists a type IV social contract $z \in Z$ such

that $z \in E_{IV}(z)$.

#

4.3.3.5   <u>THEOREM</u>:   Every type IV social system has a type IV social

contract.

<u>Proof</u>:   For each $\alpha \in A$ define the map E_{IV_α} (as in the last

proof) by

$$E_{IV_\alpha}(k,x) = t_\alpha(\{k\} \times \underline{A}(k,x)) \times \alpha(k_\alpha, x^\alpha).$$

By the upper semi-continuity of each α, $\underline{A}$ is usc, so that each

E_{IV_α} is usc, as each t_α is so by hypothesis. As $\alpha(k_\alpha, x^\alpha)$ is

non-empty, compact and convex for each $k_\alpha \in CQ(X_\alpha)$, $x^\alpha \in X^\alpha$,

$\alpha \in A$, it follows that $\underline{A}(k,x)$, hence, $\{k\} \times \underline{A}(k,x)$, has the

same properties for each $(k,x) \in Z$. Since t is a convex

and usc process, it follows that each t_α has a convex and

compact graph $\Gamma(t_\alpha) \subset Z \times CO(X_\alpha)$, as both Z and $CQ(X_\alpha)$ are

compact (and convex). Thus, $t_\alpha(\{k\} \times \underline{A}(k,x))$ is non-empty,

compact and convex, for each $(k,x) \in Z$, as it is the set

$$\pi_{CQ(X\alpha)}(\Gamma(t_\alpha) \cap (\{k\} \times \underline{A}(k,x) \times CQ(X_\alpha))).$$

Hence, $E_{IV_\alpha}(z)$ is non-empty, compact and convex in Z, for

each $z \epsilon Z$ and $\alpha \epsilon A$. As E_{IV} is nothing but the map defined

by $E_{IV}(z) = \bigcap_{A} E_{IV_\alpha}(z)$, 4.3.3.1 yields the result that there

exists a type IV social contract $z \epsilon Z$ such that $z \epsilon E_{IV}(z)$.

i #

4.3.4 Compactness and Convexity Results for Dynamic Contractual Sets

In this section it is shown that the contractual sets,

i.e., sets of social contracts, shown to be non-empty in

the last section are all compact, and that the contracutal set

consisting of type IV social contracts is also convex for a social

system which is both type III and type IV.

4.3.4.1 <u>THEOREM</u>: Let C stand for the contractual set shown to be

non-vacuous in any one of the theorems 4.3.3.2-3, 4.3.3.4.1-2

and 4.3.3.5 above. Then C is compact.

<u>Proof</u>: Let E by any one of the evolutions E_i (i.e.{ I, II, III,

IV }). Then E is usc, as each E_α was shown to be so (α ϵ A).

Then the graph

$\Gamma(E)$ $= \{(z, z') | z \epsilon Z, z' \epsilon E(z)\}$

is compact as a closed subset of the compact $Z \times Z$. Denoting

the diagonal $\{(z,z') | z=z' \epsilon Z\}$ by Δ, Δ is compact and C

is nothing but

$$C = \pi_Z(\Gamma(E) \cap \Delta),$$

hence compact.

#

4.3.4.2 THEOREM: Let S be a social system which is both type III and type IV, and let C be the set of type IV social contracts of S. Then C is convex.

Proof: As Z is convex, so is the diagonal Δ of Z × Z. Hence it suffices to show that the graph $\Gamma(E_{IV})$ is convex, as C = $\pi_Z(\Gamma(E_{IV}) \cap \Delta)$. It is obvious that $\Gamma(E_{IV})$ is convex if the graph $\Gamma(\underline{A})$ is convex, for the graph $\Gamma(t)$ is convex by hypothesis. Furthermore, $\Gamma(\underline{A})$ is convex if the graph $\Gamma(\alpha)$ of each behavor $\alpha \in A$ is convex. It is tedious but straightforward to show that $\Gamma(\alpha)$ is convex, using the hypothesis that $\ddot{w}_\alpha$ is linear ($\alpha \in A$).

#

5. EXTENSIONS AND APPLICATIONS

This chapter will identify a selected number of directions in which the framework and theory of the previous chapters can fruitfully be extended, also indicating application areas.

One large extension is into probabilistic social systems, occupying the first section. In that section a particular notion of behavior as a probability measure on a sigma-field of actions is developed, the numerical representability of preferences on sets of such behaviors is discussed, and the notion of probabilistic social system constructed.

The next section proposes a couple of axioms as necesaarily satisfied by a causal relation, and applies the resulting notion, of an event 'inducing' an event, by building on it a notion of power in probabilistic social systems. The result is compared with Dahl's [1957] concept of power and the importance of equilibrium methods for power analysis is pointed out.

The attraction and stability properties of equilibrium sets and cores is the topic of the next section. The required concepts are presented, as borrowed and modified, from the theory of dynamical systems.

Finally, the study is closed by a discussion of a number of extensions and applications particularly relevent for the management of organizations. Organizations are defined, optimal incentive problems posed, the choosing of incentive schemes related to the

act of legislation, and the relevance of equilibrium methods for all of these clarified. Three main types of control, remunerational, socializational and informational, are illustrated. Extending the detail in defining impression functions, it is shown how information-systemic elements can be incorporated into the model. Finally, the direction of multi-level social systems is pointed to as an area into which the model can be extended.

.5.1   Towards the Analysis of Probabilistic Social Systems

Along lines earlier suggested in [Sertel, 1969 a, b],
probabilistic social systems will now be developed as an important
extension of the social systems so far studied.   The next section,
5.2., illustrates one of the motivating reasons for studying
probabilistic social systems.

In 5.1.1 the notion of a behavior is particularized to that
of a probability measure on a sigma-field of actions.   5.1.2
settles matters pertaining to the measurable numerical representa-
bility of preferences on behavior spaces when the preferences are
originally specified on a sure action (a set of "sure prospects").
Then 5.1.3 finally assembles probabilistic social systems on the
basis of this groundwork.

5.1.1 Action and Behavior

The notion of 'behavior' introduced in 2.1 and used up to here is extremely general and abstract. In this section a more particular and concrete version of that notion will be constructed, founded on a certain notion of 'action'. This construction will, in turn, serve as a foundation for the treatment of "probabilistic social systems", after certain questions relating to the representation of preferences are dealt with in the next section. This and the next two sections will thus give an extension of part of the framework offered in (Sertel,1969 a).

The term 'individual' and phrases such as "things which an individual can do" will be formally undefined here; they are to be understood in the natural language sense. This prepares the ground for what follows.

5.1.1.1 <u>Definition</u>: The sure action $_o\theta_J$ of an individual j is the set of all mutually exclusive things which j can do. A sigma-field θ_j of subset of $_o\theta_j$ will be called a <u>sigma-field of actions</u> (or, for short, an <u>action field</u>) of j iff it contains the finest partition of $_o\theta_j$:

$$\{ \theta_j = \{\omega_j\} \, \big| \omega_j \, \varepsilon \, _o\theta_j \} \quad \theta_j .$$

There will always be assumed to be a unique non-trivial action field θ_j with which an individual is associated. A

subset $\theta_j \subset {}_o\theta_j$ will be called an __action__ of j iff $\theta_j \in \Theta_j$.

5.1.1.2   __Definition:__ A probability measure p_j: $\Theta_j \to [0,1]$ defined

on the action field Θ_j of an individual j will be called

a __behavior__ of j. The set P_j of all behaviors of j will be

called the __behavior space__ of j.

5.1.1.3   __Definition:__ A __collectivity__ is an ordered pair $<W,J>$,where

$W = \{P_j | j \in J\}$ is the family of behavior spaces P_j of the

individuals $j \in J$, and where J is a non-empty collection

of individuals.

5.1.1.4   __Remark:__ Since each action field is non-trivial, each behavior

space is non-empty. Compare 5.1.1.3 and 5.1.1.5 with 2.1.2-3.

5.1.1.5   __Definition:__ The __sure joint action__ of a collectivity

$<W, J>$ is the product ${}_o\theta_J = \prod_J {}_o\theta_j$ of the sure actions

${}_o\theta_j$ ($j \in J$). The __sigma-field of joint actions__ (or, for

short, __joint action field__) of $<W, J>$ is the product

$\Theta_J = \prod_J \Theta_j$ of the action fields Θ_j ($j \in J$). A subset

$\theta_J \subset {}_o\theta_J$ is called a __joint action__ of $<W, J>$ iff $\theta_J \in \Theta_J$.

5.1.1.6   __Definition:__ A (joint) probability measure p_J: $\Theta_J \to [0,1]$

defined on the joint action field Θ_J of a collectivity

$<W, J>$ will be called a __joint behavior__ or __state__ of $<W, J>$.

The set P_J of all such measures will be called the <u>joint</u>

<u>behavior space</u> of <u>state space</u> of <W,J>.

5.1.1.7   <u>Definition</u>: The <u>collective behavior space</u> of a collectivity

<W, J> is the product $P = \prod_J P_j$ of the behavior spaces

$P_j (j \in J)$.   A function $p: J \to \bigcup_J P_j$ in P is called a <u>collective</u>

<u>behavior</u> of <W,J> iff $p(j) \in P_j$ $(j \in J)$, i.e., iff $p \in P$. In

this case, $p(j)$ is also denoted by $p_j = p(j) \in P_j$. (Cf.2.1.2-3).

5.1.1.8   <u>Remark</u>:   The distinction between <u>joint</u> behaviors (or states)

and <u>collective</u> behaviors of a collectivity is crucial. To

the probabilist it will already have been clear that a behavior

is simply a certain marginal of a joint behavior and a collective

behavior is simply a specification of all such marginals for

a joint behavior.   Thus, there is a unique collective behavior

specifying these marginals of a joint behavior, whatever

joint behavior is given. Specifying a collective behavior,

however, determines either not more than one joint behavior

or not less than the cardinality of-the-continuum joint

behaviors. (This is so, for, if two joint behaviors have the

same specified marginals, then a continuum of convex

combinations also satisfy this condition, i.e., the set

of joint behaviors consistent with a collective behavior

is convex).   That there exist a joint behavior given a

collective behavior is governed, of course, by the

satisfaction of the Kolmogoroff consistency conditions.

(Kolmogoroff, 1933). (See also (Kingman and Taylor, 1966)

and (Parthasarathy, 1967).)

5.1.2 The Representation of Preference

In the social systems dealt with up to the present chapter, no mention was made of how it was that the utility functions used were guaranteed to exist as numerical representations of (complete) preference orderings on the sets in question. This is because all of the properties which were assumed, at one stage or another, to hold for these functions were properties which have been proved in the literature to be assumable without any loss of generality when certain conditions are met, and these sufficient conditions happened always to be satisfied whenever needed.

Main reference in the literature is to the representation theory of Debreu (1954) and of Herstein and Milnor (1953). For all of the results developed so far, the utility functions were assumed to have certain continuity and convexity properties. The domains of definitions for these functions were always compact and convex. Assuming the necessary and sufficient conditions demonstrated by Herstein and Milnor and presented below, convexity of the domain guarantees the existence of a real-valued linear function preserving the order of the given preferences. If the domain is topologized by the order of the preference relation, this function is easily seen also to be continuous.

Thus, all that was ever assumed so far can be seen to be assumable, using the convexity of the set ordered by preference

and the results of Herstein and Milnor. In certain cases,
Debreu's theory could have been used as an alternative,
e.g., when the domain was compact in a metric space, hence
satisfying the second axiom of countability (see Debreu's
Theorem 2).

In building toward probabilistic social systems, studied
in the next section, where the behavior spaces are of the special
kind introduced in the last section, matters of representation
of preferences are less straightforward; hence, the present
section. The exact questions which are addressed here will be
stated shortly, after some minimal preparation.

Let a collectivity $<W, J>$ be given. Fix attention to a
specific individual $j \in J$, and let $_o\theta_J \times R$ be completely
ordered by a preference relation $\leq j$ summarizing j's preferences
between elements of this set.

The first question addressed now is the following:
Under what conditions does there exist a function

$$v_j : {}_o\theta_J \times R \to R$$

such that

$$u_j(p_J, r) = \int_{{}_o\theta_J} v_j(\omega, r) dp_J(\omega)$$

exists for all $(p_J, r) \in P_J \times R$, and, such that, identifying each

$z \in {}_o\theta_J$ with the degenerate $p_J^z \in P_J$ assigning 1 to the set

$\{z\} \in \theta_J$ and 0 to the rest of θ_J, the equivalence

$$u_j(p_J^z, r) \geq u_j(p_J^{z'}, r) \text{ iff } (z,r) \geq_j (z',r')$$

holds for all $(z,r), (z',r') \in {}_o\theta_J \times R$?

This question will be answered by use of a result due to Herstein and Milnor.

Herstein and Milnor [1953] have demonstrated a necessary and sufficient triplet of conditions for the existence of a real-valued, linear function on a set M ordered completely by a relation $\leq_j$, such that the function preserves this order. These conditions are:

(1) M is a "mixture set";

(2) for all a, b, c, $\in$ M, the following sets are closed:

$$\{\lambda \in [0,1] | \lambda a + (1-\lambda)b \geq_j c\} ;$$

$$\{\lambda \in [0,1] | c \geq_j \lambda a + (1-\lambda)b\} ;$$

(3) if a, b, c $\in$ M with $a =_j b$ (where, for all d, e $\in$ M, 'd $=_j$ e' denotes 'd $\leq_j$ e and e $\leq_j$ d'), then

$$\frac{1}{2} a + \frac{1}{2}c =_j \frac{1}{2}b + \frac{1}{2}c$$

A "mixture set" here is a generalization of a convex set in a real vector space.

Now this result will be used to obtain some reasonable conditions under which the desired sort of function v_j will exist. First, it is assumed that the set $_o\theta_J$ is convex in a real vector space. It follows that $_o\theta_J \times R$ is convex, hence a "mixture set". Next, assume that the conditions (2) and (3) are met by $M = _o\theta_J \times R$. Then there exists a linear real-valued function v_j: $_o\theta_J \times R \rightarrow R$ preserving $\leq_j$. It will now be seen that this function v_j has all the desired properties, after a few more assumptions are made.

Topologize $_o\theta_J \times R$ with the order topology $\leq_j$. That is, topologize $_o\theta_J \times R$ with the coarsest topology for which

$$\{\{b \, \epsilon \, _o\theta_J \times R | b \geq_j a\}, \{b \, \epsilon \, _o\theta_J \times R | a \leq_j b\} | a \, \epsilon \, _o\theta_J \times R\}$$

is a family of closed sets. (Note: This is not the order topology defined by Eilenberg (1941). The definition of Eilenberg would correspond to the quotient space where elements of an equivalence class according to the order are not distinguished, even though they may be distinct in $_o\theta_J \times R$. Although the quotient space of the space defined here is always T_1 (in fact, T_2, i.e., Hausdorff), this is not true of the space itself, as can be seen from the fact that if $b \neq a$ but $b \geq_j a$ and $a \geq_j b$, then there is no neighborhood of b(a) of which a (b) is not an element. The course of definition chosen here is motivated by the need,

in the present context, to keep distinctness of points distinct from non-equivalence of points). It is obvious that v_j is then continuous. Hence, v_j is continuous whenever $_o\theta_J \times R$ has a topology finer than the order topology of $\leq_j$.

The final assumption to be made concerning $_o\theta_J \times R$ is that $_o\theta_J$ is a quasi-compact T_1 (topological) subspace of a linear topological space. The first consequence of this comes from the fact that $_o\theta_J \times R$ is T_1, i.e., that the singleton subsets $\{a\}$ $_o\theta_J \times R$ are closed. The mentioned consequence of this is that v_j is continuous. This is so, for a T_1 topology on $_o\theta_J \times R$ = M is finer than the order topology. To see that a T_1 topology is finer than the order topology, take a, c $\in$ M such that c $\notin$ $\{b \in M | b \geq_j a\}$ and note that the complement of $\{b \in M | b \geq_j a\}$ is open in the T_1 topology by the fact that c $\geq_j$ a implies c to be distinct from a, whereby there is a nbd of c (in the T_1 topology) which does not meet $\{b \in M | b \geq_j a\}$.

Since $_o\theta_J \times R$ fails to be quasi-compact, however, it does not follow from the continuity of v_j that v_j is bounded. This is a serious deficiency, as boundedness coupled with continuity of v_j would guarantee its integrability with respect to each $p_j \in P_J$, as desired. It is this deficiency to which a remedy will now be sought.

Let $\ddot{\gamma}_j : {}_o\theta_J \rightarrow R$ represent a typical element of the set of linear real-valued functions on $_o\theta_J$ for which $\ddot{\gamma}_j({}_o\theta_J) \subset B$, where B is a fixed (bounded) closed interval $[r_1, r_2]$.

(The functions $\ddot{\gamma}_j$ are analogous to the (interpreted) incentive

functions $\ddot{g}_\alpha : X \to R$ of the previous chapters; the symbol

$\ddot{g}_j$, however, is reserved here for a different but closely related

function.) By the fact that $\ddot{\gamma}_j$'s are restricted to have their

ranges contained in the bounded set B, it is being assumed that

no infinitely large rewards or punishments are distributed in

the social systems about to be considered.

From this assumption an important consequence will now be

obtained. First to be noted is that each $\ddot{\gamma}_j$ is obviously bounded.

Actually, this follows from the fact that $_o\theta_J$ is quasi-compact

and that, by its linearity, $\ddot{\gamma}_j$ is continuous, as $\ddot{\gamma}_j\left(_o\theta_J\right)$ is

therefore quasi-compact, hence bounded as a subset of R. But,

also, ν_j may now be taken to be a function $\nu_j : _o\theta_J \times B \to R$, since

its restriction to $_o\theta_J \times B$ is all that matters in the social system

which the present development is obviously heading toward.

The result of this is that the function

$$u_j(p_J, r) = \int_{_o\theta J} \nu_j(\omega, r) \, dp_J(\omega)$$

is now well-defined, having imposed the constraint $r \in B$, as

ν_j is continuous and bounded on $_o\theta_J \times B$, hence integrable with

respect to each $p_J \in P_J$. The property desired for ν_j with

respect to degenerate elements of P_J is satisfied by its linearity.

Thus, u_j now serves as a "utility function" for the individual j,

the domain of definition being $P_J \times B$, where P_J is the joint

behavior space of the collectivity $<W, J>$.

Before turning to the next section, a natural continuation of the above development to be settled here is deriving incentive functions for the individuals in J. Again, attention will be fixed to the individual $j \in J$.

Define the function $\ddot{g}_j : P_J \to R$ by

$$\ddot{g}_j(P_J) = \int_{0}^{\theta_J} \ddot{\gamma}_j(\omega) \, dp_J(\omega).$$

Since $\ddot{\gamma}_j$ is continuous and bounded, $\ddot{g}_j$ is well defined. Now it is clear that this function, $\ddot{g}_j$, will play the role of $\ddot{g}_\alpha$ (in the previous chapters) for the behavor j of the coming section.

5.1.3   Probabilistic Social Systems

Building on the groundwork provided in the last two
sections, it is now possible to define probabilistic social
systems as a variety of the general social system introduced at
the outset of the study.   This will provide an extension of a formal
entity earlier introduced in (Sertel, 1969 a).   The mentioned
formal entity (although called a "collectivity" therein) was
a finite-personnel version of what is about to be defined here
as a probabilistic social system, the behavior spaces being
closed geometric simplexes in Euclidean space, with incentive
and interpretation schemes missing from the specification while
an impression scheme was present.   The social system specified
was shown in that and an accompanying study (Sertel, 1969 b)
to have a non-empty compact and convex set of equilibria.
The distinguishing characteristics of such equilibria in
comparison to the equilibrium shown by Nash (1950, 1951) to
exist for games of a similar specification consisted of two
components.   Firstly, impression functions were not explicit
- or they were implicitly assumed to be identity maps -
in Nash's specification.   Secondly, the equilibria of Nash were
collective behaviors, while those of Sertel were joint behaviors,
referring to the terminology and very important distinction
(see 5.1.1.8) introduced in 5.1.1.

The importance of being able to deal with joint behaviors
is most clear when an extension of the present study to an
investigation, e.g., of the core is considered. For
the joint randomization of the actions of members has to
be considered as in the very nature of a coalition when the
notion of behavior is probabilistic. This point must therefore
be emphasized as crucial also to any political analysis, if
any, which is to benefit from the methods suggested by the
present study, since it may be expected to be essential especially
in political analysis to be realistic about the workings of
coalitions. The theory of power suggested later in this
study is anticipated in the last remark.

A motivation for defining and studying probabilistic
social systems should be easily extractible from the components
above. With minor effort, relying on the previous sections,
a definition will soon be formalized. The major portion of this
effort has to be directed toward constructing a number of
functions. This is now taken up.

5.1.3.1   <u>Definition-Notation</u>:   Let P_J be the joint behavior space of

a collectivity $<W, J>$.

Denote $_o\Theta^j = \prod\limits_{J \{j\}} {}_o\Theta_i$, denote the product sigma-field

$\prod\limits_{J \{j\}} \Theta_i$ by Θ^j, and denote the set of probability measures

p^j_J: $\Theta^j \to [0,1]$ by P^j_J. Define the function $\mu_j : P_J \to P_j$ as follows:

$P_{J_j} = \mu_j(p_J)$ has the property that, for any $\theta_j \in \Theta_j$,

$$p_{J_j}(\theta_j) = p_J(\theta_j \times {}_o\theta^j);$$

$\mu_j(p_J)$ will be called the $\underline{j\text{-marginal}}$ of p_J. Define the function $\mu^j: P_J \to P_J^j$ as follows:

$p_J^j = \mu^j(p_J)$ has the property that, for any $\theta^j \in \Theta^j$,

$$p_J^j(\theta^j) = p_J({}_o\theta_j \times \theta^j);$$

$\mu^j(p_J)$ will be called the $\underline{j\text{-exclusive marginal}}$ of p_J.

5.1.3.2 $\underline{\text{Definition}}$:   A function $u_j: P_J \times R \to R$ will be called a $\underline{\text{utility function}}$ of j.   A function $h_j: P_J^j \to P_J^j$ will be called an $\underline{\text{impression function}}$ of j. A function $g_j: P_J \to R$ will be called an $\underline{\text{incentive function}}$ for j.   A function i_j assigning to each incentive function g_j for j an incentive function $\breve{g}_j = i_j(g_j)$ for j will be called an $\underline{\text{interpretation}}$ $\underline{\text{function}}$ for j.   A non-empty collection $D_j \subset [P_j]$ of non-empty subsets $d_j \subset P_j$ will be called a $\underline{\text{feasibility space for }j}$, and each element $d_j \in D_j$ will be called a $\underline{\text{feasibility}}$ for j. Let $\{d_j | j \in J\}$ be a family of feasibilities d_j, one for each $j \in J$.   The set $d \subset P_J$ of all joint behaviors p_J such that,

for each $j \in J$, $\mu_j(p_J) \in d_j$ will be called a <u>collective feasibility</u> for J. The set of all collective feasibilities for J will be called the <u>collective feasibility space</u> of J and denoted by D. A mapping $t_j : P_J \times D \to D_j$ will be called a <u>feasibility transformation</u> of j. The mapping $t: P_J \times D \to D$ defined by

$$t(p_J, d) = \{q_J \in P_J \mid \mu_j(q_J) \in t_j(p_J, d) \text{ for all } j \in J\}$$

will be called the <u>collective feasibility transformation</u> of J.

5.1.3.3 <u>Definition</u>: Let $\langle W, A \rangle$ be a collectivity, let

$U = \{u_\alpha \mid \alpha \in A\}$ be a family of utility functions,

$H = \{h_\alpha \mid \alpha \in A\}$ a family of impression functions,

$G = \{g_\alpha \mid \alpha \in A\}$ a family of incentive functions,

$I = \{i_\alpha \mid \alpha \in A\}$ a family of interpretation functions,

$T = \{t_\alpha \mid \alpha \in A\}$ a family of feasibility transformations,

and let A be a family of self-indexed mappings

$$\alpha : P_A \times D_\alpha \to [P_\alpha] \quad \text{defined by}$$

$$\alpha(p_A^\alpha, d_\alpha) = \{p_\alpha \in d_\alpha \mid \ddot{w}_\alpha(p_\alpha * p_A^\alpha) \geq \sup_{d_\alpha} \ddot{w}_\alpha(q_\alpha * p_A^\alpha)\} ,$$

$(q_\alpha \in d_\alpha)$ where the operator $*$ and the function $\ddot{w}_\alpha$ are

defined below. Then the ordered seven-tuplet S =
<W, U, H, G, I, T, A> will be called a <u>probabilistic social</u>
<u>system</u>.

5.1.3.4   <u>Definition</u>: Let P_j and P_J^j be as in 5.1.3.1. The binary
operation * is defined by

$$P_j * P_J^j (\theta_j \times \theta^j) = P_j(\theta_j)\, \ddot{P}_J^j(\theta^j), \quad (\theta_j \in \Theta_j, \theta^j \in \Theta^j).$$

5.1.3.5   <u>Definition</u>: Denote $i_\alpha(g_\alpha) = \ddot{g}_\alpha$ and

$$u_\alpha(p_\alpha * h_\alpha(p_A^\alpha), \; i_\alpha(g_\alpha)\,(p_\alpha * h_\alpha(p_A^\alpha)))$$

$$= \ddot{u}_\alpha(p_\alpha * h_\alpha(p_A^\alpha), \; \ddot{g}_\alpha(p_\alpha * h_\alpha(p^\alpha{}_A)))$$

$$= \ddot{v}_\alpha(p_\alpha * h_\alpha(p_A^\alpha))$$

$$= \ddot{w}_\alpha(p_\alpha * p_A^\alpha).$$

The derived function $\ddot{w}_\alpha$ will be called the <u>effective utility</u>
<u>function</u> of α.

5.1.3.6 <u>Research Problems</u>: The investigation of the non-emptiness

and other properties of equilibrium and Pareto sets and the

core of probabilistic social systems in general is a research

problem not undertaken here. A start toward this is the

already mentioned study (Sertel, 1969 a, b) of a finite and

static case - not reproduced here - but the results of that

study can probably be generalized to quite a degree. The

relation between the equilibrium set, Pareto set and core may

be especially interesting in the case of probabilistic

social systems. The reason for making this conjecture is

the point made earlier in this section (preceding 5.1.3.1).

-concerning the coordinated randomization of members' actions

for coalitions in the case of probabilistic social systems.

5.2   Toward a Framework for the Analysis of Power

With the probabilistic social systems of the previous section in mind, the present section now turns to a topic which is central to political analysis, namely, power. A notion of one event (action) 'inducing' another will be introduced. Being related to the concept of causality between events, it will be used to examine the formal constituents of power relations between agents. The main data governing such power relations will be derived from the joint behaviors of the social system supposed. The non-transiency of these relations will be seen to depend on whether the social system is at equilibrium.

The present discussion will get further by, rather than starting from scratch, agreeing in principle with the intuitive idea of power that guided Dahl: "α has power over β to the extent that he can get β to do something that β would otherwise not do" [Dahl, 1957, pp. 202-3] (Dahl's notation is different than the one used here.) And, taking this as a point of departure, there is no visible route which both promises to lead toward a fruitful destination and succeeds in completely by-passing the subject of causality. For that reason,it will enhance the exposition to agree from the outset on a minimal but workable committment as to when a given event will be considered to be a cause of another given event. The question of what are fruitfully to be considered as necessary and sufficient conditions for such a causal relation to be said to exist between

a given pair of events is too deep to be addressed here. For
what is ahead, however, it will be important to agree on some
necessary conditions. The choice of such conditions will be guided
by the objective of economizing on commitment subject to the con-
straint of obtaining something that is reasonably workable and non-
vague. In doing so, the reader will be left free to add any
further axioms which appear to be desirable. A whole host of
questions concerning time-precedence, contiguity, etc., will thus
be left to the reader to exercise his personal metaphysics with
regard to. Differing from Suppes [1967], the weakest necessary
condition is chosen as expressed in the following definition.

5.2.1   Definition: Let $\$$ be a sigma-field of events and p: $\$ \to [0,1]$
a probability measure defined on $\$$, A relation $\kappa \subset \$ \times \$$
will be said to satisfy the **first axiom of causality with respect**
to p iff the following condition is satisfied:

> (E,F) ε κ only if there exists (an event)
> E' ε $\$$ such that $p[E'] > 0$ and
> $p[F|E'] < p[F|E]$.

5.2.2   Remark: What is required by the above axiom is quite minimal.
If (E,F) ε κ is ever to be read as "the event E is a cause of
event F when the probability judgement p is made," it is
being demanded that there be some event E' such that F is less

likely, according to that judgement, when it is known that

E' is the case than when E is known to be the case. If this is

not taken as an axiom, then it would have to be considered as

reasonable to say "E causes F although F is at least as likely to

occur under any circumstances which are at all likely (judged

to have non-zero probability measure) as under the circumstance of

E".

It is obvious that the event E' which is demanded must be

distinct from E, i.e., a different subset of the sure event on

which $ is a sigma-field. This is so, for otherwise $p[F|E']$ would

be equal to $p[F|E]$. What is very important to recognize, however,

is that E' is not required to be the complement of E^C of E. In

fact, E' can be an event in $ which is a (proper) subset of E and

still satisfy the requirement imposed, while neither E^C nor,

indeed, any event in $ which is a subset of E^C, need satisfy the

requirement.

It is precisely this which constitutes the fundamental

disagreement of the axiom chosen in 5.2.1 with what Suppes

[1967] takes to be minimal as a necessary condition. (Reference

is to his Definition 1 of "prima facie cause", page 11 of

Suppes' Chapter 5.) By concentrating on the complement of E,

Dahl appears in his framework to have anticipated Suppes' point

of departure, although this appearance may be due to the vagueness

of Dahl's notation in this regard, which in turn may be due to

the fact that he did not work with the clear notion of a sigma-field of events and may have been unaware of the distinction between two events being complements and two events being distinct.

Now an additional and final axiom will be introduced to capture the notion of "directionality" in a "causal relation".

5.2.3   <u>Definition</u>: Let \$ and p be as in 5.2.1. A relation $\kappa \subseteq \$ \times \$$ will be said to satisfy the <u>second axiom of causality with respect to p</u> iff the following condition is satisfied:

$$(E,F) \in \kappa \text{ only if } p[F|E] \geq P[E|F].$$

To summarize 5.2.1 and 5.2.3, the following will be useful.

5.2.4   <u>Definition</u>: Let p be a probability measure defined on a sigma-field \$ of events, and let E, F $\in$ \$ be two events. E will be said to <u>p-induce</u> F (denoted as $E \xrightarrow{p} F$) iff there exists a relation $\kappa \subseteq \$ \times \$$ such that κ satisfies the first and second axioms of causality and $(E,F) \in \kappa$.

It may be conjectured, as it was by the present author, that $E \xrightarrow{p} F$ and $F \xrightarrow{p} G$ implies $E \xrightarrow{p} G$ (where E, F, G $\in$ \$, for some sigma-field \$ and where p: $\$ \rightarrow [0,1]$ is a probability measure). The conjecture is false, as the counterexample kindly provided by

P. R. Kleindorfer (personal communication), presented below, demonstrates.

5.2.5 <u>Proposition</u>: (The relation) $\underset{p}{\rightarrow}$ need <u>not</u> be transitive.

<u>Proof</u>: Let $\$$ be the power set of $\{a, b, c, d\}$, and let $p: \$ \rightarrow [0,1]$ be a probability measure such that $p(\{a\}) = p(\{b\}) = p(\{c\}) = p(\{d\}) = 1/4$. Then

$$1/2 = p(\{a, b\}|\{b,c\}) > p(\{a,b\} \{b,c,d\}) = 1/3,$$

and

$$p(\{a, b\}|\{b,c\}) = p(\{b,c\}|\{a,b\}) = 1/2,$$

so that

$$\{b,c\} \quad \underset{p}{\rightarrow} \{a, b\} .$$

Also,

$$1/2 = p(\{b, c\}|\{c, d\}) > p(\{b, c\}|\{a, b, d\}) = 1/3$$

and

$$p(\{b,c\}|\{c,d\}) = p(\{c, d\}|\{b,c\}) = 1/2,$$

so that

$$\{b, c\} \underset{p}{\rightarrow} \{a, b\} \quad \text{and} \{c, d\} \underset{p}{\rightarrow} \{b, c\} .$$

However, it is <u>not</u> the case that $\{c, d\} \underset{p}{\rightarrow} \{a, b\}$, since $p(\{a, b\}|\{c, d\}) = 0$ contradicts that the <u>first</u> axiom of causality holds for any $\kappa \subset \$ \times \$$ such that $(\{c, d\}, \{a, b\}) \in \kappa$.

#

5.2.6.　Remark: It has long·been recognized that "the causal arrow"

cannot be regarded as the 'copula' of logical implication,

notably for the reason that the latter contraposes while the

former does not. Another difference between "the causal arrow"

and "the implication arrow" is revealed by 5.2.5, if $\underset{p}{\rightarrow}$ is accepted

as a "causal arrow". (This is so, for containment '$\subset$' is transitive.)

Note also that a sure event can never be p-induced, for

there cannot exist a non-null subset conditional upon which its

probability is less than unity. Thus, e.g., with reference

to 5.2.5, for any $E \varepsilon \$$, $E \underset{p}{\rightarrow} \{a, b, c, d\}$ implies that $p(E) = 0$,

which is a contradiction: the first axiom of causality effectively

prevents one from saying, for an event, which happens anyway,

that it is "caused" by some event.

Completely sacrificing the formal development of any interesting

mathematical consequences from the axioms or definitions introduced,

the promised application of the above to the topic of power will now

be pursued. The context of what follows is a proababilistic social

system with a personnel A of typical behavors $\alpha \varepsilon$ A. For any

coalition $B \subset A$, $\Theta_B = \underset{B}{\Pi} \Theta_\beta$ is the joint action field of B, and

Θ_β is shorthand for $\Theta_{\{\beta\}}$; P_B is the behavor space of B, consisting

of all joint behaviors $p_B = \Theta_B \rightarrow [0, 1]$ of B.

The following definition formalizes two key concepts.

5.2.7   Definition: Let A be the personnel of a probabilistic social

system S, and let B, B' $\subset$ A be coalitions. For any joint

behavior p_A of A, the power relation $M(B, B'; p_A)$ is the set

$$M(B,B'; p_A) = \{(\theta_B, \theta_{B'}) | \theta_B \in \Theta_B, \theta_{B'} \in \Theta_{B'}, \theta_B \underset{p_A}{\rightarrow} \theta_{B'} \}.$$

The power relation $M(B', B; p_A)$ is defined by replacing B with

B' and B' with B in the last expression. The power structure

of S subject to p_A is the set $M(S; p_A) = \{M(B, B'; p_A) | B, B' \in [A]\}$

of all power relations $M(B, B'; p_A)$ between coalitions B, B' $\subset$ A.

The power relations defined above particularize to "interpersonal"

relations when only singleton coalitions are considered (Cf.(Frey,

n.d; p. 17).) They, as well as $M(S; p_A)$, can be "quantified" in

the fashion now to be indicated.

5.2.8   Definition: Let $M(B, B'; p_A)$ be a power relation in a power

structure $M(S; p_A)$. Define the function m for each ordered pair

$(\theta_B, \theta_{B'})$ with $\theta_B \in \Theta_B$ and $\theta_{B'} \in \Theta_{B'}$ by

$$m(\theta_B, \theta_{B'}) = \begin{cases} p_A[\theta_{B'} | \theta_B], & \text{if } \theta_B \underset{p_A}{\rightarrow} \theta_{B'} \\ 0, & \text{otherwise.} \end{cases}$$

The function m will be called the numerical representation of $M(B,$

$B'; p_A)$. The set of numerical representations of all elements

of $M(S; p_A)$ will be called the numerical representation of $M(S; p_A)$.

5.2.9   <u>Remark</u>: The definanda of 5.2.7-8 are all determined by one datum

alone, namely, the joint behavior p_A of S.

5.2.10 <u>Remark</u>: A brief comparison of the function m with Dahl's "amount

of power" [1957, p. 205] is in order. Although the probability

measure he uses is not a joint behavior and his notion of action

is not clear, with some harmless change in notation, Dahl's

definition of the <u>amount of power of B over B"</u>, <u>with respect to</u>

<u>the response</u> $\theta_{B'}$, <u>by means of</u> θ_B sets this amount equal to

$$\Delta(\theta_B, \theta_{B'}) = \text{Prob } [\theta_{B'} | \theta_B] - \text{Prob } [\theta_{B'} | \theta_B^c]$$

If the probability p_A is used, then Δ becomes more easily

comparable with m. In that case, m becomes the counterpart

here of Δ in Dahl's framework. The two are very different functions

of course, since the formalization of the underlying notion of

power here fundamentally differs from that of Dahl. In the

present framework $p_A[\theta_B' | \theta_B^c]$ has no particular significance

in the obtaining of the number m $(\theta_B, \theta_{B'})$, as it has no special

role in determining whether or not $\theta_B \underset{p_A}{\rightarrow} \theta_{B''}$. It is important

to note that m will detect some cases where $\theta_B \underset{p_A}{\rightarrow} \theta_{B'}$ by

assigning m$(\theta_B, \theta_{B'}) > 0$ while Δ will fail to do so. In fact

Δ may assign $\Delta(\theta_B, \theta_{B'}) < 0$ while $\theta_B \underset{p_A}{\rightarrow} \theta_{B''}$, so that m$(\theta_B, \theta_{B'}) > 0$.

Hence from the standpoint of the development here, Dahl's

measure has to be classified as misleading. It is remarkable that

two formalizations of the same intuitive notion (recorded at the

outset), should give such disagreeing results. The fact is, however, that the notion formalized here is actually the following: B has power over B' with respect to inducee action $\theta_{B'}$, by means of inducing acting θ_B, and subject to p_A, iff $\theta_B \xrightarrow{p_A} \theta_{B'}$: the amount of this power is the probability $p_A[\theta_{B'}|\theta_B]$ of the inducee action conditional upon the inducing action.

In the above definitions and remarks concerning power, an arbitrary joint behavior p_A of the social system was used in computing all the necessary probabilities. The importance of that joint behavior being an equilibrium point is clear, if the power relations and power structure are to be considered as non-transient. For if the very fact that a certain power structure (or p_A) holds leads to its being altered, as is the case for any non-equilibrium p_A, then the power structure (or p_A) in question is transient and not a regularity. It is important, therefore, whether there exists an equilibrium p_A, for, if there does, then the associated power structure is an equilibrium power structure. That there does exist an equilibrium p_A for certain probabilistic social systems was shown in [Sertel, 1969 a]. Generalizations of that result and the investigation of the attraction and stability properties of equilibrium sets and cores appear clearly to promise an important bearing on political analysis.

5.3 Towards the Analysis of Attraction and Stability

Given that the equilibrium set or Pareto set or core of a social system is non-empty, two important and related types of question arise concerning these sets: attraction and stability. The tools for investigating these topics are to be found in the theory of dynamical systems. Naturally, this is also where the notions themselves of attraction and stability are developed, so it is there that one has to turn in order to see precisely what these are. A few preliminary definitions of this theory will be presented here to crystallize the required concepts. Then it will be shown how a social system may be looked upon as a dynamical system, so that the theory of the latter may be applied to the former. Finally, some discussion will follow.

Possibly the most prominent author on attraction, as well as the originator of the notion of weak attraction, is Bhatia (1966). The definitions to follow, however, are borrowed from another prominent author, Szegö(1968). They are slightly modified in harmless fashion to relate most directly to social systems as an area of application. Because of this, some of the terms have been changed, in order to avoid confusion. Notably, the notion of a dynamical system has been modified and the term "evolutionary system" attached to the result.

For the following definitions, let X be a locally compact Hausdorff space, and denote the set of non-negative integers by Z, taking the order topology on Z. A bar across the top will indicate topological closure.

5.3.1 Definition: A evolutionary system is an ordered triplet $\langle X, Z, E \rangle$, where

(5.3.1.1) $E: X \times Z \to X$ is an usc point-to-set mapping;

(5.3.1.2) $E(x, 0) = x$ $\quad (x \in X)$;

(5.3.1.3) $E(E(x,m),n) = E(x, m+n)$ $\quad (x \in X; m, n \in Z)$.

5.3.2 Definition: The future of a point $x \in X$ is the set $F(x) = E(\{x\} \times Z)$.

5.3.3 Definition: The limit set of a point $x \in X$ is the set

$$L(x) = \bigcap_{y \in F(x)} \overline{F(y)}.$$

5.3.4 Definition: Let $M \subset X$ be compact.

(5.3.4.1) $A^-(M) = \{x \mid L(x) \cap M \neq \emptyset \}$

is called the <u>region of weak attraction</u> of M. M is called

a <u>weak attractor</u> iff $A^-(M)$ is a nbd of M.

(5.3.4.2) $A^+(M) = \{x \mid \emptyset \neq L(x) \subset M \}$

is called the <u>region of attraction</u> of M. M is called an

<u>attractor</u> iff $A^+(M)$ is a nbd of M.

5.3.5   <u>Definition</u>:   Let $M \subset X$ be compact.

(5.3.5.1)   M is <u>stable</u> iff, for every nbd V of M, there exists a nbd

U of M such that $F(U) \subset V$.

(5.3.5.2)   M is <u>asymptotically stable</u> iff it is a weak attractor and

stable.

The above definitions will now be interpreted from the viewpoint

of their application to social systems. The space X is to be

interpreted as the domain of an <u>evolution</u>, so that, depending

on the social system in mind, X will be either simply the collective

(or joint) behavior space, or it will be the product of the

collective (or joint) behavior space with the collective feasibility

space. In each of the cases where the contractual set was proved

to be non-empty in this study, X was compact, hence satisfying

the requirement that it be a locally compact Hausdorff space.

Now the mapping E can be related to the evolution of a social

system. Denoting the latter by E^1, equate $E^1(x) = E(x,1)$. Then

E^1 is the one-time-application of E. The condition 5.3.1.2

states that if E is applied zero times, then nothing changes.

If it is applied two times,then $E(x, 2) = E(E(x,1), 1) = E^1(E^1$

$(x))$, and so on. This is clearly consistent with the idea of

an evolution. As to E being usc, it has to be remarked that

E^1 was usc in every case where it was shown to have a fixed point.

From the fact that E^1 is usc, it follows that, defining

$$E^0(x) = E(x, 0),$$

$$E^n = E^1(E^{n-1}) \quad (n = 1, 2, \ldots),$$

E^n is usc for any $n \in Z$.

Now let V be a nbd of $E(x, m)$. To show that E is usc, one

needs to show the existence of nbd U of (x,m) such that $E(U) \subset V$.

Note that $E(x, m) = E^m(x)$ and that $\{m\}$ is a nbd of m. Since

E^m is usc, there exists a nbd N of x such that $E^m(N) \subset V$. Then

$U = N \times \{m\}$ is a nbd of (x, m) such that $E(N \times \{m\}) = E^m(N) \subset V$.

Hence, E is usc, from the assumption that E^1 is so. Thus, as

long as usc evolutions are used, as done in this study, to

establish the non-emptiness of a contractual set, it is harmless
to assume 5.3.1.1. This concludes the justification for
5.3.1 as a whole.

Regarding the set M used in the definitions 5.3.4-5, notice
that it can be interpreted as a contractual set (or core, etc.)
as long as compactness is guaranteed for the latter. In the case
of the sets proved to be non-empty in this study, the requirement
is met.

The idea intended to be communicated by the present section
is that the theory of dynamical systems may offer the tools required
for the attraction and stability analysis of the equilibrium set
and core of a social system. The modifications with which the
definitions above were presented amount to incorporating the case
where E is a point-to-set mapping, as corresponding to the fact
that an evolution E^1 is, in general, of this nature.
Without some rather stringent assumptions (in the nature
of strict quasi-concavity for certain restrictions of effective
utility functions) the evolution of a social system will usually
not be a point-to-point mapping. In this case, the usual Liapunov
or simpler methods of stability analysis are inapplicable, so that
some remedy has to be sought. The discussion above is the result
of some groping in that direction.

5.4   Towards the Planning and Control of Organizations

The purpose of this section is to indicate selected extensions

and applications of the above framework and theory which bear

especially on the management of social systems. The section has to

be selective for the same sort of reasons that would force one to be

so if one were listing the uses of addition and multiplication. It

has to be selective also for the reason that, after a point, it

is more fruitful to do than to talk about the doing, to extend

and to apply rather than to endlessly converse on where and how to

extend and to apply.

"Management", at least for the present discussion, is the

guessing of what is an achievable "best" and the seeing to it

that such a best is achieved. So, with no great loss in paraphrasing,

it is planning and control. The guessing of what is achievable and

best, i.e., planning, is a matter of knowing what are reachable

points of the universe, having criteria of goodness for those

points, and, last but not at all least, having a framework and

accompanying methods, tools of analysis, to actually select a

point. And by the framework and method of analysis is not meant

as much an optimizing algorithm as is meant a way of thinking,

modelling, faithfully abstracting essentials and simply

representing, in a fashion that is perhaps communicable to some

optimizing algorithm.

Supposing that a best achievable point is known, the actual

seeing to it that the point is approximated in reality is a matter

of controlling the system whose performance is in question and which

performance is one of the reachable."points" in this abstract

discourse. The ability to control this system in turn depends on

what variables - "knobs, buttons, and levers", as it were - can

be set, and on knowing how the system responds to the various

values at which these can be set.

When the system in question is a social system, as it

always is in any non-trivial management problem, and when it

is a large system, there is really no way to manage but to work

with a highly abstract model of it. This is not to say that

one cannot manage or improve the performance of a given hospital,

school, business organization, football team or economy without

such abstraction. It is to say, however, that - as Polya is

known to have remarked - a trick will work once, and it is a method

that works the next time.

It is the obtaining of such methods of management which is

addressed as an application here. It will be taken as granted that

the system to be managed is a social system, its performance

depending on the behavior of the system and that behavior being

a concise way of expressing the behaviors of the members, or ,

in general, the coalitions.

It will be assumed that there is some criterion or objective

functional which numerically represents the performance of the

system as a function of its behavior. If there is no such guide

for comparing one behavior of the system with another behavior, then it is not possible to guess what is a "best" behavior; so, to talk of managing the system, such a guide has to be assumed to exist. In reality, much of the difficulty of management may be due to the absence of such a guide in clearcut form. But there is nothing that can be done about that here. If the objective is too vague, it will not be possible to discriminate good management from bad anyway.

The nature of this objective functional can be expressed with the example of an economy. Think of a gross sort of "production function" which shows national income as a function of levels of various activities. Supposing that national income is the better the larger, i.e., that it is a true measure of performance, and looking at the mentioned "activities" as behaviors of one sort of another - or as aggregations of and decomposable into such - what one has is an objective functional of the kind that will be supposed.

It is an opportune moment to define an organization $\Omega = \langle S, q \rangle$ as a social system S together with an objective functional $q:X \to R$, where X is either, as usual, the collective, or the joint behavior space of S. For the sake of simplicity, assume that the personnel $N = \langle 1, \ldots, n \rangle$ of S is finite.

Often, when an organization $\Omega = \langle S, q \rangle$ is specified, q is a "gross" and not a "net" objective functional, in a sense that will now be seen. Consider the case of a business organization

and let q represent revenue net of all expenses but wages and

salaries. The incentive functions g_j (j ε N) express amounts,

in money units for the example being considered, given to the

various members of the personnel. Summing up, one obtains the

payroll function $p = \sum_N g_j$, so that

$$p(x) = \sum_N g_j(x) \qquad\qquad (x\varepsilon X)$$

represents the additional payroll expense which has to be

deducted from q(x) to compute the profit $f(x) = q(x)-p(x)$ as

a function of the behavior x of the organization Ω.

Suppose now that one seeks the "optimal" incentive scheme

$g = (g_1, \ldots, g_n)$. For a profit-maximizing concern, the function

sought is g* such that, among all incentive schemes, g* maximizes

profit. But this is not such a clear statement yet, for the

behavior x of Ω depends on the incentive scheme imposed, and there

may be more than one possible way in which the system S behaves for

a given g. Furthermore, the set of these behaviors corresponding

to g maybe a large set. Worse, there may be no equilibrium behavior

when g is imposed, the behavior of S cycling around in that set of

behaviors associated with g. It is not possible in general then

to write x as a function $\psi(g)$ of g and then write $f = f(\psi(g))$ to search

for an optimal, i.e., f-maximizing, incentive scheme g*.

If there exists an equilibrium behavior $x \in \psi(g)$, however, then things are different. Suppose it is known then that there is a fairly wide class Γ of incentive schemes such that for every $g \in \Gamma$, the system S has an equilibrium. Such a class Γ is identified by the equilibrium existence theorems of this study for a variety of social systems S. Suppose that a fairly wide subset of Γ were identified to be a set of incentive schemes for which the equilibrium set has further nice properties, such as being an attractor and being stable. We already know that the equilibrium set is compact for each of the cases where it was proved in this study to be non-empty. Suppose that further studies, extensions of the present one, teach us how to find a subset Γ' of Γ for which the equilibrium sets are all very small, and stable attractors. Then $\psi(g)$ for each $g \in \Gamma'$ can be represented, for all practical purposes, by a single point $x \in \psi(g)$.

Now, returning to the original problem of optimizing the incentive scheme, write

$$(\text{Max}) \quad f(\psi(g))$$

$$\text{s.t} \quad g \in \Gamma'$$

as a well-defined optimization problem. Problems of this sort have been considered by Kriebel and Lave [1969]. A particularly interesting case is that of the "constant-share finite organization", where $g_j = \lambda_j q$, $\lambda = (\lambda_0, \lambda_1, \ldots \lambda_n)$ being a non-negative vector in E^{n+1} with $\sum_{j=0}^{n} \lambda_j = 1$, determining $f = \lambda_0 q$ as the profit

function. Assuming negative definite quadratic effective utility

functions $\ddot{w}_j$ and similar q, preliminary results have been obtained

by P. R. Kleindorfer and this author.

The above constitutes one illustration of how one would

usefully proceed from the present study in obtaining results

central to the management of organizations. The basic ideas behind

what was just illustrated arose during the writing of a paper on

organizational structures (Zannetos and Sertel, 1971).
This should be taken as an indication that they will be followed up.

While on the topic of incentive schemes, it is opportune

to mention how this relates to the topics of regulation and

legislation as specific instances of the general topic of controlling

social systems.

No great imagination is required to see that the idea of

an incentive scheme is an idealization of the notion of a

rule, regulation or law in a social context. The sanctions behind

laws are not always real-valued, but usually real vector-valued.

For example, a sentence of 18 months imprisonment and a $35,000

fine can be pronounced together. Suspensions of licences, etc.,

are also sanctions. So, in general, incentives are not real-

valued. But this is not a tremendous blow to the model of a social

system presented here, for vector-valued incentives can be

incorporated with no serious trouble, except for some tedium in

some of the proofs. In order not to complicate the model any

further, so that its main features stand out more clearly,

incentives were represented as real numbers. Recognizing all this,
however, the business of legislation is easily seen to be truly
a managerial activity. The choice of incentive scheme for the
profit-maximizing organization considered above and the making
of laws are essentially the same sort of thing.

How do the existence and various properties of equilibrium
or core points relate to the topic of legislation? Perhaps the
easiest way to communicate how is by means of an example. Take
the case of a typical unsuccessful "rural reform", in a backward
and strongly feudal region, which redistributes land and illegalizes
all taxes paid by peasants to landlord. The mere passing of the
law makes little differences to reality, for the system tends
right back to its original equilibrium, if ever it is disturbed
in the first place. To prevent what is intended to be prevented,
an incentive scheme has to be found under which the undesired
status-quo is left outside the equilibrium set. To make sure it
really works, the equilibrium set induced by the legislation has
to be an attractor and stable, so that the behavior of the
system is attracted toward this set and, once attracted, stays
in that vicinity.

Successful legislation requires, therefore, an equilibrium
analysis of the social system, whether this is based on strong
social intuition or mathematics. Often it is not possible to
alter the status-quo without a re-socialization of the personnel.

Consider the case where one is illegalizing head-hunting in the
Phillipines, polygamy in a traditional Islamic community, racial
discrimination in South Africa, or slavery in earlier America.
To the people whose behavior has to be altered, there is nothing
wrong with their present behavior, in fact, what is asked of them
seems ridiculous or even immoral to them. There is little
that can be done by legislation, except for that legislation which
affects the socialization process, giving new values to new vintages
of entrants into the personnel, while the older vintage dies.

Black-marketeering and smuggling are typically behaviors
which cannot be prevented except by readjustment of the relative
price vector, for it is the fact that they belong to the equilibrium
set of the present price vector that accounts for their presence.
Adjusting the price vector, of course, often defeats the purpose
of making certain goods and services unavailable in the first place,
so this constitutes no way out. The idea of illegalizing certain
production and trade activities and imposing sanctions tries to
add new "price tags" of possible imprisonment, etc., to the usual
one of pence and piastres, and seeks thus to make the dealing
in the markets in question unattractive. It succeeds to the
extent that the elongated new "price vector" moves the equilibrium
set away from those collective behaviors in which the undesired
behaviors are components.

Another example of a case where some equilibrium analysis
is needed is the prevention of fraud in an accounting system. The usual

rule is well-advised: make it necessary to form as large coalitions

as possible for fraud to become undetectable. Requiring that at

least two people coalesce for any fraud to succeed is the usual

extent to which this rule is carried. The extreme application would

be to expel all fraud from the core. That, of course, may require

a bit more than intuition to successfully do.

The example of the accounting system, however, is interesting

in a different sense than the earlier examples. For the method

of fraud-prevention here is also information-systemic rather than

purely incentive-sanction based. For it is already illegal

and severely punishable to cheat. The key is to yield the undesirable

behavior detectable or observable whenever it becomes possible,

ié., unblocked.

All the above hints at three main means of social control:

remunerational (via incentive schemes), socializational (via

alterations in utility schemes) and informational. The last mentioned

can be explored a bit further. The impression functions of the

framework used in this study can be decomposed into two functions.

Take a typical impression function h_α : $X^\alpha \to X^\alpha$. Let $\delta_\alpha : X^\alpha \to X^\alpha$

be a function called the data function reporting to α. Look at

the function $\delta: X \to \Pi X^\alpha$ defined by $\delta(x) = \Pi_A \delta_\alpha(x)$. This function
is appropriately considered as an information system. The data

flowing through it, the filters and aggregation imposed, are all part

of the information system design. Now respecify the old impression

function h_α as follows:

$$h_\alpha(x^\alpha) = \tilde{h}_\alpha(x^\alpha, \delta_\alpha(x^\alpha)). \qquad (x^\alpha \in X^\alpha)$$

In the new form, α sees two things, x^α and $\delta_\alpha(x^\alpha)$. Of these $\delta_\alpha(x^\alpha)$ is what he is told about x^α. From all this, $\tilde{h}_\alpha$ obtains an impression $h_\alpha(x^\alpha)$. This impression is the α-exclusive behavior which α then bases his choice of behavior on.

Now if $\tilde{h}_\alpha$ is given as a parameter of α, h_α can be influenced by altering δ_α. That is to say, h, the impression scheme, can be altered by altering δ, the information system. This alters the choice of behavior, the equilibria, and so on.

The above indicates a further direction in which the model used in this study can be extended, and identifies a further form of informational control, that is, control via the information system. It is easy to see how the specification of the interpretation scheme could be modifed also in similar fashion.

Finally, the above discussion can now be used to indicate a further extension of the model, towards multi-level social systems. For consider now the fact that there are many people who already realize that a social system can be controlled in the ways described above, each imposing an incentive scheme, socialization process or information system. It is possible to view the behaviors of these controlling agents as points in suitable function spaces, and the agents themselves as behavors in a "second-level" social

system, affecting the behavior of the underlying social system. It is possible, therefore, to carry out an analysis of this second level as one did of the first, and to investigate how the two levels relate. For example, how do the respective equilibria relate? "Hierarchical" social systems thus become an extension of the ones considered here.

This concludes the present discourse on how to fruitfully extend the present model. Now is the time to begin the investigations indicated.

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BIOGRAPHY

Murat Sertel received his B.A. (Economics), from Robert College, Istanbul ("High Honors", 1963). Worked for the State Planning Organization, Ankara, in the Summers of 1962, 1963. Joined the Department of Economics and Statistics, Middle East Technical University, Ankara,1963 as 'asistan'. He declined a Fullbright Scholarship in favor of the 1964 five-year Rockefeller Fellowship awarded by the American Colleges of Istanbul for doctoral studies in Decision Theory. He enrolled for postgraduate studies at Oxford University, 1964, spending 1964-1965 at Balliol College. He was awarded a two-year Nuffield Studentship, 1965; spent 1965 - 1966 at Nuffield College, receiving B. Phil. (Economics), 1966. Began doctoral program at the Sloan School of Management, Massachusetts Institute of Technology, 1967, after a summer as special technical assistant there, also holding a research assistantship in the Summer of 1968. He completed general examinations in June, 1969, to join the Managerial Information, Planning and Control Group, Sloan School of Management, as an instructor.

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